

INNOVATION PROCESSES

Y. A. Rozanov



$$H_t(\xi) = \bigoplus_{j=1}^M H_t(X_j)$$



$$\bigcap_t H_t(\xi) = 0$$



$$E_t = \int_{-\infty}^{\infty} e^{i\lambda t} d\tilde{E}_{\lambda}$$

INNOVATION PROCESSES

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The importance of the role played by innovation processes in filtering theory has been recognized only recently, even though the basic idea had been postulated by Kolmogorov in his early contributions. In this book, Professor Yuriy Anatol'yevich Rozanov examines the problem of constructing innovation processes in a very general setting.

The author, Senior Scientist at the Steklov Mathematical Institute of the Academy of Sciences of the USSR and one of the leading mathematicians on the faculty of Moscow University, won the Lenin Prize in Science for his contributions to probability and random processes theory. Three of his earlier books, *Stationary Random Processes*, *Infinite-dimensional Gaussian Distributions*, and *Introductory Probability Theory*, have been translated into English and published in the U.S. Among other published translations of Professor Rozanov's important contributions to the field is *Probability Theory: Basic Concepts, Limit Theorems, Random Processes* co-authored with Y. V. Prohorov.

Innovation Processes presents a new approach to the basic problems which arise in prediction, filtering,

(continued on inside back flap)

INNOVATION PROCESSES

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Translation Editor

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TRANSLATION EDITOR'S PREFACE

The importance of the role played by innovation processes in filtering theory has been recognized only recently, even though the basic idea had, of course, been prepared by Kolmogorov in his early work. Here Professor Yu. A. Rozanov, one of the pioneers in multi-dimensional filtering theory and a well-known Soviet specialist in stochastic processes, examines the problem of constructing innovation processes in a very general setting.

Much of the work he presents is based on his own extensive research and unifies much scattered material for the first time. Using innovation processes as the basic theme, he develops some of the fundamental aspects of Gaussian processes such as:

Regularity and multiplicity (including the infinite dimensional case);

Wold decomposition and factorization;

Equivalence and orthogonality.

The presentation brings fresh insight into these areas.

He has clearly succeeded in creating a succinct, systematic account of this important subject, which should be of considerable interest to both pure and applied mathematicians.

A. V. Balakrishnan

Chapter I

INNOVATION PROCESSES AND CANONICAL REPRESENTATIONS

§1. Introduction

Let $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$ be a multidimensional random process with the components $\xi_i(t)$, $i = 1, 2, \dots, m$ (m can be infinite). Let us assume that all the values $\xi_i(t)$ have zero mean $E\xi_i(t) = 0$ and finite second moments $E|\xi_i(t)|^2 < \infty$. We shall denote by $H_t(\xi)$ the closed (in the meansquare) linear hull of all the values $\xi_i(s)$, $t_0 < s \leq t$, and regard $H_t(\xi)$ as a subspace of the Hilbert space of all the complex valued random variables η , $E|\eta|^2 < \infty$, with the scalar product $\langle \eta_1, \eta_2 \rangle = E\eta_1 \bar{\eta}_2$. Here we do not distinguish between variables equal with probability one.

In linear problems of random process theory, such as those of linear prediction and filtering, the corresponding subspace $H_t(\xi)$ represents "a collection of data" the "observer" has available at time t to estimate a variable η , $E|\eta|^2 < \infty$ and the projection of this variable into the subspace $H_t(\xi)$ is the best *mean square* estimate.

Denote by P_t the operator of orthogonal projection onto $H_t(\xi)$ and introduce the space

$$H(\xi) = \overline{\bigcup_{t_0 < t < T} H_t(\xi)}.$$

The family of projection operators P_t , $t_0 < t < T$, on $H(\xi)$, as well as the family of the corresponding subspaces

$$H_t(\xi) = P_t H(\xi), \quad t_0 < t < T,$$

increasing with t :

$$H_s(\xi) \subseteq H_t(\xi) \quad \text{for } s \leq t,$$

will be the main subject of our investigation. The principal problem is to characterize the temporal evolution of subspaces $H_t(\xi)$.

To indicate our general approach to the solution of this problem, we shall discuss a one-dimensional random process $\xi(t)$, $t_0 < t < T$ with uncorrelated increments. We shall assume this process to be left continuous:

$$\lim_{h \rightarrow +0} \xi(t-h) = \xi(t),$$

and set $F(t) = \mathbf{E} |\xi(t)|^2$. We shall call the monotone non-decreasing left continuous function $F(t)$, $t_0 < t < T$ a *structural* function. The corresponding subspace $H_t(\xi)$ consists of all variables η representable as the stochastic integral

$$\eta = \int_{t_0}^T c(t) d\xi(t),$$

where the complex-valued function $c(t)$, $t_0 \leq t < T$, satisfies the condition

$$\int_{t_0}^T |c(t)|^2 dF(t) < \infty.$$

If we consider the Hilbert space $\mathbf{C}$ of all such functions $c(t)$, $t_0 \leq t < T$, with the scalar product

$$\int_{t_0}^T c_1(t) \overline{c_2(t)} dF(t),$$

and in this space the subspaces $\mathbf{C}_t$ of all functions vanishing outside the interval $[t_0, t]$, we can see that the subspaces $H_t(\xi)$ are unitarily isomorphic to the appropriate subspaces $\mathbf{C}_t$, since $H_t(\xi)$ consists of all variables of the form*

$$\eta = \int_{t_0}^t c(s) d\xi(s),$$

where $\int_{t_0}^t |c(s)|^2 dF(s) < \infty$. Roughly speaking, the family of the subspaces $H_t(\xi)$, $t_0 < t < T$ is arranged exactly as the family of the subspaces $\mathbf{C}_t$, $t_0 < t < T$, whose evolution as t increases, looks fairly obvious and can be completely characterized by an appropriate function $F(t)$.

In considering the general case we shall assume the space $H(\xi)$ separable, and the family $H_t(\xi)$, $t_0 < t < T$ left-continuous:

$$\lim_{h \rightarrow +0} H_{t-h}(\xi) = H_t(\xi). \quad (1.1)$$

*Here and further on in the integrals of the form $\int_s^t$ integration is carried out over the interval $[s, t]$, closed to the left and open to the right, with

$$\int_{t_0}^t d\xi(s) = \xi(t), \quad \int_s^t d\xi(u) = \xi(t) - \xi(s)$$

and

$$\int_{t_0}^t dF(s) = F(t), \quad \int_s^t dF(u) = F(t) - F(s), \quad t_0 < s < t < T.$$

These conditions can be satisfied if, for example, the initial random process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$ is left continuous:

$$\lim_{h \rightarrow +0} \xi_i(t-h) = \xi_i(t), \quad i = 1, \dots, m.$$

The question of the evolution of the subspaces $H_t(\xi)$, $t_0 < t < T$ for an arbitrary random process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$ would be considered solved if we can find uncorrelated processes $X_j(t)$, $j = 1, \dots, M$, $t_0 < t < T$ with uncorrelated increments, such that the subspaces $H_t(X)$ corresponding to the multi-dimensional process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$ coincide with the subspaces $H_t(\xi)$:

$$H_t(X) = H_t(\xi), \quad t_0 < t < T. \quad (1.2)$$

Indeed, then one could represent the subspaces $H_t(\xi)$ as the orthogonal sum

$$H_t(\xi) = \bigoplus_{j=1}^M H_t(X_j), \quad t_0 < t < T,$$

where each of the mutually orthogonal families $H_t(X_j)$, $t_0 < t < T$, is described, as mentioned above, by means of an appropriate structural function

$$F_j(t) = \mathbf{E} |X_j(t)|^2, \quad t_0 < t < T; \quad j = 1, \dots, M. \quad (1.3)$$

We shall call the random process $X(t) = \{X_j(t)\}_1^M$ with the orthogonal components $X_j(t)$, $t_0 < t < T$ ($X_j(s) \perp X_k(t)$ for all s, t with $j \neq k$), each of which represents a process with uncorrelated increments satisfying condition (1.2), an *innovation process* for the random process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$.

Note that under our assumption (1.1), according to which

$$\lim_{h \rightarrow +0} H_{t-h}(X) = H_t(X),$$

the innovation process is left-continuous.

Obviously, an innovation process always exists. Its components $X_j(t)$, $t_0 < t < T$ can be constructed, for instance, in the following manner. Choose some element $x_1 \in H(\xi)$ and define the process with uncorrelated increments $X_1(t) = P_t x_1$, $t_0 < t < T$ (where P_t denotes the projection operator onto the subspace $H_t(\xi)$). If $H(X_1) \neq H(\xi)$, then we shall select some element $x_2 \in H(\xi)$ orthogonal to $H(X_1)$, and define the process with the uncorrelated increments $X_2(t) = P_t x_2$, $t_0 < t < T$. Since the subspace $H(X_1)$ and its orthogonal complement are invariant under the projections P_t , $t_0 < t < T$, the process $X_2(t)$, $t_0 < t < T$ will be uncorrelated with the process $X_1(t)$, $t_0 < t < T$. If $H(X_1) \oplus H(X_2) \neq H(\xi)$, then we shall select the next element x_3 , orthogonal to the subspaces $H(X_1)$ and $H(X_2)$, and define the process with uncorrelated increments $X_3(t) = P_t x_3$, $t_0 < t < T$, which will be uncorrelated with $X_1(t)$ and $X_2(t)$, $t_0 < t < T$. Choosing in a similar manner the successive elements x_k :

$$x_k \perp \bigoplus_{j=1}^{k-1} H(X_j),$$

we shall finally exhaust the whole space $H(\xi)$ having obtained the mutually uncorrelated processes

$$X_j(t) = P_t x_j, \quad t_0 < t < T; \quad j = 1, \dots, M, \quad (1.4)$$

with uncorrelated increments (their number M can be infinite). Obviously, $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$ is an innovation process for $\xi(t)$, $t_0 < t < T$, since $P_{t+h} x_j - P_t x_j \perp H_t(\xi)$ with $h > 0$ and the system of variables $P_s x_j$, $t_0 < s \leq t$, $P_{t+h} x_j - P_t x_j$, $h > 0$; $j = 1, \dots, M$, is complete in $H(\xi)$, so that the system of variables $P_s x_j$, $t_0 < s \leq t$; $j = 1, \dots, M$ is complete in $H_t(\xi)$, and, therefore,

$$H_t(X) = H_t(\xi), \quad t_0 < t < T.$$

If we introduce the Hilbert space $\mathbf{C}$ of all vector functions $\mathbf{c}(t) = \{c_j(t)\}_1^M$, $t_0 \leq t < T$, with components $c_j(t)$,

$t_0 \leq t < T$, satisfying the condition

$$\int_{t_0}^T \sum_{j=1}^M |c_j(t)|^2 dF_j(t) < \infty, \quad (1.5)$$

with scalar product

$$\int_{t_0}^T \sum_{j=1}^M c_{1j}(t) \overline{c_{2j}(t)} dF_j(t); \quad \{c_{1j}(t)\}_1^M, \quad \{c_{2j}(t)\}_1^M \in \mathbf{C},$$

where $F_j(t) = \mathbb{E} |X_j(t)|^2$, $t_0 < t < T$,

are the corresponding structural functions of the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$ [see (1.3)], then the space $H(\xi)$ can be described as an aggregate of all variables of the form

$$\eta = \int_{t_0}^T \sum_{j=1}^M c_j(t) dX_j(t), \quad (1.6)$$

where $\{c_j\}_1^M \in \mathbf{C}$, and, obviously,

$$P_t \eta = \int_{t_0}^t \sum_{j=1}^M c_j(s) dX_j(s), \quad (1.7)$$

since $H(X) = H(\xi)$ and $H_t(X) = H_t(\xi)$. It is seen that the subspace $H_t(\xi) = P_t H(X)$ consists of all variables of the form

$$\eta = \int_{t_0}^t \sum_{j=1}^M c_j(s) dX_j(s), \quad \{c_j(s)\}_1^M \in \mathbf{C},$$

and, by the unitary isomorphism $\eta \leftrightarrow \{c_j(s)\}_1^M$, one can say that the family $H_t(\xi)$, $t_0 < t < T$, is of the same type as that of the

family of the subspaces $\mathbf{C}_t$, $t_0 < t < T$, each of which is formed by means of the functions $\{c_j(s)\}_1^M \in \mathbf{C}$, vanishing for $s > t$ (for appropriate t).

Formula (1.6) applied to the process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$, yields a so-called *canonical* representation

$$\xi_i(t) = \int_{t_0}^t \sum_{j=1}^M c_{ij}(t, s) dX_j(s), \quad i = 1, \dots, m, \quad (1.8)$$

which allows us to describe the projection operators P_t , $t_0 < t < T$, in the space $H(\xi)$:

$$P_u \xi_i(t) = \int_{t_0}^u \sum_{j=1}^M c_{ij}(t, s) dX_j(s), \quad i = 1, \dots, m, \quad (1.9)$$

with $t_0 \leq u < t < T$.

From the practical point of view the problem of finding methods for constructing linear transformations connecting the processes $\xi(t)$ and $X(t)$, in particular, for constructing the canonical representation (1.8), is most crucial.

From the theoretical point of view the questions of how to define the type of the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, or how to compare it with a certain given type knowing the correlation function of the initial process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$, are of major interest.

Interest in this problem arose with the work of H. Cramér* who showed that the innovation process for the one-dimensional continuous process $\xi(t)$, $t_0 < t < T$, can be, generally speaking, arbitrary. Starting from the prescribed innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, of an arbitrary structure (with arbitrary structural functions $F_j(t) = \mathbf{E}|X_j(t)|^2$, $j = 1, \dots, M$), the corresponding process $\xi(t)$, $t_0 < t < T$, can be constructed as follows.

*H. Cramér, Stochastic processes as curves in a Hilbert space, Theory of Probability and Application, IX, 2 (1964), 193-204.

Let $\Delta_j, j = 1, \dots, M$ be non-overlapping measurable sets on an interval (t_0, T) , $\bigcup_{j=1}^M \Delta_j = (t_0, T) \pmod{0}$, such that the intersection of each set with any interval (s, t) , $t_0 < s < t < T$ has positive measure. Define the integrable process $\dot{\xi}(t)$, $t_0 < t < T$, having chosen the function $\sigma(t) > 0$ properly and set $\dot{\xi}(t) = \sigma(t) X_j(t)$ with $t \in \Delta_j, j = 1, \dots, M$. Consider $\xi(t) = \int_{t_0}^t \dot{\xi}(s) ds$. Clearly $H_t(\xi) \subseteq H_t(X)$ at all t , $X_j(s)$ coinciding with the derivative $\xi'(s)$ for almost all $s \in \Delta_j$, and since $\xi'(s) \in H_t(\xi)$ for $s \leq t$, and the sets Δ_j are everywhere dense in the interval (t_0, T) , and the processes $X_j(s)$ are left continuous, we have that $X_j(s) \in H_t(\xi)$ for all $s \leq t$, so that $H_t(X) \subseteq H_t(\xi)$ and, therefore, $X(t)$ is an innovation process for the continuous process $\xi(t)$, $t_0 < t < T$.

Here is one example of the sets $\Delta_1, \Delta_2, \dots$.

Let K be the Cantor set of the interval $I = (t_0, T)$ obtained by rejecting successively "adjacent" intervals $I_{1/2}; I_{1/4}; I_{3/4}; I_{1/8}; I_{3/8}; I_{5/8}; I_{7/8}$; etc., selected according to the measure

$$\mu(K) = \mu(I) - \sum_a \mu(I_a) = m,$$

where $m > 0$ is given ($m \leq \mu(I)$).

On each adjacent interval I_{α_1} , of "the first class" let us take our Cantor set K_{α_1} with measure m_{α_1} , obtained by rejecting from I_{α_1} the adjacent intervals I_{α_1, α_2} (α_2 runs through the binary rational sequence $1/2, 1/4, 3/4, \dots$). Set $K^{(1)} = \bigcup_{\alpha_1} K_{\alpha_1}$. Next, on each ad-

jacent interval $I_{\alpha_1 \alpha_2}$ of "second class" let us take again the Cantor set $K_{\alpha_1 \alpha_2}$ with measure $m_{\alpha_1 \alpha_2}$, obtained by rejecting from $I_{\alpha_1 \alpha_2}$ the adjacent intervals $I_{\alpha_1 \alpha_2 \alpha_3}$. Set $K^{(2)} = \bigcup_{\alpha_1, \alpha_2} K_{\alpha_1 \alpha_2}$. Continuing this pro-

cedure successively for all the adjacent intervals $I_{\alpha_1 \dots \alpha_n}$ and assuming $K^{(n)} = \bigcup_{\alpha_1, \dots, \alpha_n} K_{\alpha_1 \dots \alpha_n}$, we shall obtain the sequence

of non-overlapping sets $K^{(0)}=K, K^{(1)}, K^{(2)}, \dots, K^{(n)}, \dots$. Assume, in

addition, that the union $\bigcup_{n=0}^{\infty} K^{(n)}$ has unit measure: $\sum_{n=0}^{\infty} \mu(K^{(n)}) =$

$\mu(I)$ (this can be achieved by properly choosing the measure $m_{a_1 \dots a_n} > 0$ of the particular Cantor components $K_{a_1 \dots a_n}$).

Recall that the Cantor set $K_{a_1 \dots a_n}$ is nowhere dense in $I_{a_1 \dots a_n}$ and any interval $\delta_n \subseteq I_{a_1 \dots a_n}$ has a non-empty intersection with a certain adjacent interval of the subsequent class $I_{a_1 \dots a_n a_{n+1}}$. Take any interval $(a, b) \subseteq I$ and consider the imbedded interval (a', b') , $a' = a + \varepsilon$; $b' = b - \varepsilon$. As noted above, (a', b') has a non-empty intersection with a certain adjacent interval I_{a_1} ; set $\delta_1 = (a', b') \cap I_{a_1}$. The interval $\delta_1 \subseteq I_{a_1}$ has a non-empty intersection with a certain adjacent interval $I_{a_1 a_2}$; set $\delta_2 = \delta_1 \cap I_{a_1 a_2}$. It is clear that the interval (a', b') , with any n , has a non-empty intersection with a certain adjacent interval $I_{a_1 \dots a_n}$. But $\mu(I_{a_1 \dots a_n}) \rightarrow 0$, and for sufficiently large n (for which $\mu(I_{a_1 \dots a_n}) < \varepsilon$) the corresponding interval $I_{a_1 \dots a_n}$ intersecting with the interval $(a + \varepsilon, b - \varepsilon)$, is contained entirely in the primary interval (a, b) . Thus, any interval $(a, b) \subseteq I$ for all sufficiently large n contains a certain adjacent interval of the " n -th class" $I_{a_1 \dots a_n}$, and, consequently, it contains as well a set $K^{(n)} \subseteq I_{a_1 \dots a_n}$ of positive measure $\mu(K^{(n)}) = m^{(n)}$. Decompose the sequence of positive whole numbers $n = 0, 1, \dots$ into a countable number of non-overlapping subsequences $n_{11}, n_{12}, \dots; n_{21}, n_{22}, \dots; \dots$ and set

$$\Delta_p = \bigcup_{q=1}^{\infty} K^{(n_{pq})}, \quad p = 1, 2, \dots$$

Clearly, any interval $(a, b) \subseteq I$ contains, for sufficiently large q , sets $K^{(n_{pq})}$ of positive measure $m^{(n_{pq})}$, so that

$$\mu\{(a, b) \cap \Delta_p\} > 0$$

for all $p = 1, 2, \dots$

§2. Structural Types and Subordinate Processes

1. *Some auxiliary statements.** Let $H_t(\xi)$, $t_0 < t < T$ be a family of monotone increasing left continuous subspaces $H_t(\xi)$ and a separable Hilbert space $H(\xi)$, and P_t , $t_0 < t < T$ be the corresponding family of projection operators projecting onto $H_t(\xi)$, $t_0 < t < T$.

Take an arbitrary element $x \in H(\xi)$ and consider the closed linear hull of all variables $P_t x$, $t_0 < t < T$, denoting it by $H(x)$. The subspace $H(x)$ consists of all variables of the form

$$\eta = \int_{t_0}^T \varphi(t) d\Phi(t), \quad (2.1)$$

where $\Phi(t) = P_t x$, $t_0 < t < T$, and the function $\varphi(t)$ satisfies the condition

$$\int_{t_0}^T |\varphi(t)|^2 dF(t) < \infty$$

($F(t) = \mathbf{E} |\Phi(t)|^2$ is the structural function of the process with the uncorrelated increments $\Phi(t)$, $t_0 < t < T$).

Let $y \in H(x)$. According to the general formula (2.1) the variable y can be represented as the stochastic integral

$$y = \int_{t_0}^T \varphi_{yx}(t) d\Phi(t),$$

Then

*Actually, these are well-known results in the theory of "spectral types." (See, for example, the survey by A. L. Plessner and V. A. Rokhlin, Spectral theory of linear operators, Uspekhi mat. nauk, I, 1(11) (1946).

$$P_t y = \int_{t_0}^t \varphi_{yx}(s) d\Phi(s), \quad t_0 < t < T,$$

and

$$G(t) = \mathbf{E} |\Psi(t)|^2 = \int_{t_0}^t |\varphi_{yx}(s)|^2 dF(s), \quad t_0 < t < T.$$

will be the structural function of the process $\Psi(t) = P_t y$, $t_0 < t < T$.

As we know the space $H(y)$ consists of all variables of the form

$$\eta = \int_{t_0}^T \psi(t) d\Psi(t) = \int_{t_0}^T \psi(t) \varphi_{yx}(t) d\Phi(t),$$

where

$$\int_{t_0}^T |\psi(t)|^2 dG(t) = \int_{t_0}^T |\psi(t) \varphi_{yx}(t)|^2 dF(t) < \infty.$$

Denote by Δ the support of the measure $dG(t)$. Clearly

$$\int_{t_0}^T \psi(t) d\Psi(t) = \int_{\Delta} \psi(t) d\Psi(t) = \int_{\Delta} \psi(t) \varphi_{yx}(t) d\Phi(t).$$

Since $dG(t) = |\varphi_{yx}(t)|^2 dF(t)$, then $\varphi_{yx}(t) \neq 0$ almost everywhere on the set Δ , and setting $\varphi(t) = \psi(t) \varphi_{yx}(t)$ for $t \in \Delta$, we obtain that the space $H(y)$ consists of all variables of the form

$$\eta = \int_{\Delta} \varphi(t) d\Phi(t), \quad (2.2)$$

where $\int_{\Delta} |\varphi(t)|^2 dF(t) < \infty$. It is useful to note that in the case

where the measures $dG(t)$ and $dF(t)$ are equivalent, $dG \sim dF$, Δ is at the same time the support of the measure $dF(t)$ and we have the equality:

$$H(y) = H(x).$$

Let us consider now some innovation processes $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, defined by the relationships of the type (1.4): $X_j(t) = P_t x_j$, $j = 1, \dots, M$, where the elements x_j are chosen so that $X_j(s) \perp X_k(t)$ for all s, t with $j \neq k$. Hence

$$H(\xi) = \bigoplus_{j=1}^M H(x_j).$$

Since each of the mutually orthogonal subspaces $H(x_j)$ is invariant with respect to the projection operators P_t , $t_0 < t < T$, for any element $y \in H(\xi)$ we have

$$P_t y = \bigoplus_{k=1}^M P_t y_k, \quad t_0 < t < T,$$

where y_k is the projection of y onto the corresponding subspace $H(x_k)$, $k = 1, \dots, M$. We know that the structural functions $G_k(t) = \mathbf{E} |P_t y_k|^2$ are absolutely continuous with respect to the corresponding functions $F_k = \mathbf{E} |P_t x_k|^2$, $t_0 < t < T$, and, consequently, the structural function $G(t) = \mathbf{E} |P_t y|^2$,

$$G(t) = \sum_{k=1}^M G_k(t), \quad t_0 < t < T,$$

is absolutely continuous with respect to the function

$$F^*(t) = \sum_{k=1}^M \sigma_k^2 F_k(t), \quad t_0 < t < T, \quad (2.3)$$

where the constants $\sigma_1, \sigma_2, \dots$ are chosen so that the series

$$\sum_{k=1}^M \sigma_k^2 \|x_k\|^2 \quad \text{converges} \quad (\text{clearly, } F^*(t) = \mathbb{E} |P_t x^*|^2 \quad \text{with } x^* = \sum_{k=1}^M \sigma_k x_k).$$

We can see that all possible "structural types" $dG(t)$ are absolutely continuous with respect to some maximal structural type $dF^*(t)$.

Let H be a certain subspace, invariant with respect to the family of projection operators P_t , $t_0 < t < T$. Call the element $x \in H$ maximal in H , if for any $y \in H$ the structural function $G(t) = \mathbb{E} |P_t y|^2$, $t_0 < t < T$, is absolutely continuous with respect to the structural function $F(t) = \mathbb{E} |P_t x|^2$, $t_0 < t < T$ ($G(t) \leq F(t)$). As was actually shown above, the maximal element $x \in H$ with the maximal structural type $dF(t)$ always exists [see (2.3)].

Let $x_1 \in H(\xi)$ be the maximal element in the space $H(\xi)$ and $F_1(t) = \mathbb{E} |P_t x_1|^2$, $t_0 < t < T$. Let x_2 be the maximal element in the subspace $H(\xi) \ominus H(x_1)$ and $F_2(t) = \mathbb{E} |P_t x_2|^2$, $t_0 < t < T$. Generally, let x_k be the maximal element in the subspace

$$H(\xi) \ominus \left[\bigoplus_{j=1}^{k-1} H(x_j) \right]$$

and

$$F_k(t) = \mathbb{E} |P_t x_k|^2, \quad t_0 < t < T; \quad k = 1, \dots, M.$$

Clearly, the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, with the components $X_j(t) = P_t x_j$, $t_0 < t < T$; $j = 1, \dots, M$, will have the structural functions $F_j(t)$, $t_0 < t < M$, ordered in the sense that

$$dF_1(t) \succ dF_2(t) \succ \dots \succ dF_M(t) \quad (2.4)$$

(i.e., each of the measures $dF_j(t)$ is absolutely continuous with respect to the antecedents $dF_1(t), \dots, dF_{j-1}(t)$).

We shall show further that the ordered structural types (2.4) are defined uniquely by the family $H_t(\xi)$, $t_0 < t < T$; more precisely, for any innovation process $Y(t) = \{Y_k(t)\}_1^N$ ($H_t(Y) = H_t(\xi)$, $t_0 < t < T$) with the ordered structural functions $G_k(t) = \mathbf{E} | \dot{Y}_k(t) |^2$, $t_0 < t < T$ [i.e., such that $dG_1(t) > dG_2(t) > \dots > dG_N(t)$],

$$\begin{aligned} N &= M, \\ dG_j(t) &\sim dF_j(t), \quad j = 1, \dots, M \end{aligned} \quad (2.5)$$

(the corresponding number M is usually called *multiplicity*).

We shall say for short that the measure $dG(t)$ is subordinate to $dF(t)$ if it is absolutely continuous with respect to the measure $dF(t)$: $dG(t) < dF(t)$; and $dG(t)$ is *orthogonal* to $dF(t)$, if these measures have non-overlapping supports: $dG(t) \perp dF(t)$.

Let $dF(t)$ be an arbitrary positive bounded measure on the interval $[t_0, T)$, subordinate to the maximal structural type $dF^*(t) \sim dF_1(t)$. There exists an element $x \in H(\xi)$ with a structural type exactly equal to $dF(t)$, i.e., such that

$$\mathbf{E} | P_t x |^2 = \int_{t_0}^t dF(s), \quad t_0 < t < T,$$

We can take, for example,

$$x = \int_{t_0}^T \left[\frac{dF(t)}{dF_1(t)} \right]^{1/2} dX_1(t),$$

where, we recall, $F_1(t) = \mathbf{E} | X_1(t) |^2$ the maximal structural function [see (2.4)]. Clearly, all the elements $y \in H(\xi)$ with the structural types $dG(t)$, subordinate to $dF(t)$, form a linear subspace which is invariant with respect to the projectors P_t , $t_0 < t < T$.

Consider the system of elements $y_1, y_2, \dots, y_n \in H(\xi)$ with the same structural type $dG(t)$, such that the subspaces $H(y_k)$ generated by the variables $P_t y_k$, $t_0 < t < T$, are orthogonal with various $k = 1, \dots, n$. We shall call the system $y_1, \dots, y_n$ maximal, if it

cannot be extended; more precisely, if there exists no element $y \in H(\xi)$, orthogonal to the subspaces $H(y_k)$, $k = 1, \dots, n$, and having $dG(t)$ as its structural type. Clearly, the maximal system exists.

Lemma 1. *If $y_1, \dots, y_m$ is the maximal system, and $y'_1, \dots, y'_n$ is some system with the same structural type $dG(t)$, then*

$$n \leq m. \quad (2.6)$$

PROOF. Suppose that the subspace $L = \bigoplus_{j=1}^m H(y_j)$ coincides with the whole space $H(\xi)$. If we introduce $Y_j(t) = P_t y_j$, $t_0 < t < T$; $j = 1, \dots, m$, then the elements $y'_1, \dots, y'_n \in L$ can be represented as

$$y'_k = \int_{t_0}^T \sum_{j=1}^m c_{kj}(t) dY_j(t),$$

where the functions $c_{kj}(t)$ satisfy the condition

$$\int_{t_0}^T \sum_{j=1}^m |c_{kj}(t)|^2 dG(t) < \infty, \quad k = 1, \dots, m.$$

Since all the elements $y'_1, \dots, y'_n$ have the same structural function,

$$G'(t) = \int_{t_0}^t \sum_{j=1}^m |c_{kj}(s)|^2 dG(s), \quad k = 1, \dots, n,$$

by the orthogonality of the subspaces $H(y'_1), \dots, H(y'_n)$ it follows that

$$(P_t y'_i y'_k) = \int_{t_0}^t \sum_{j=1}^m c_{ij}(s) \overline{c_{kj}(s)} dG(s) = 0, \quad \text{for } i \neq k,$$

and hence

$$\sum_{j=1}^m |c_{kj}(t)|^2 = 1, \quad \sum_{j=1}^m c_{ij}(t) \overline{c_{kj}(t)} = 0 \quad \text{for } i \neq k$$

for almost all t with respect to the measure $dG(t)$. It is seen that the vectors $c_k(t) = \{c_{kj}(t)\}_1^m$, $k = 1, \dots, n$, with the components $c_{kj}(t)$, $j = 1, \dots, m$, in m -dimensional vector space (with usual scalar product) form an orthonormalized system. As is well known, the number of elements of such a system does not exceed the dimension of the space, i.e., $n \leq m$.

The general case can be reduced to that examined above by means of the following. Introduce the space H of all elements whose structural types are subordinate to $dG(t)$, and consider the subspace

$$L = \bigoplus_{j=1}^m H(y_j) \text{ and its orthogonal complement } L_0 = H \ominus L. \text{ By}$$

the definition of the maximal system $y_1, \dots, y_m$ in the subspace L_0 there is no element with the structural type $dG(t)$. Hence, if $dG_0(t)$ is the maximal structural type in the invariant subspace L_0 , then $dG(t) = dG_0(t) \oplus d\tilde{G}(t)$, where $d\tilde{G}(t)$ is a certain non-zero measure orthogonal to $dG_0(t)$. Consider a new space $\tilde{H}$ —the subspace in H of all the elements with structural types subordinate to $dG(t)$. It is easily seen that since the measures $dG_0(t)$ and $d\tilde{G}(t)$ are orthogonal, the subspaces L_0 and $\tilde{H}$ with the maximal structural types $dG_0(t)$ and $d\tilde{G}(t)$ will be also orthogonal, and hence $\tilde{H} \subseteq L$,

$$\text{where } L = \bigoplus_{j=1}^m H(y_j).$$

If Δ_0 and $\tilde{\Delta}$ denote the non-overlapping supports of the orthogonal measures $dG_0(t)$ and $d\tilde{G}(t)$, then any variable $\eta \in H(y_k)$ can be represented as

$$\eta = \int_{t_0}^T \psi(t) d\Psi_I(t) = \int_{\Delta_0} \psi(t) d\Psi_I(t) \oplus \int_{\tilde{\Delta}} \psi(t) d\Psi_I(t)$$

$$(\Psi_I(t) = P_t y_I, \quad t_0 < t < T),$$

where

$$\eta_j = \int_{\Delta_j} \psi(t) d\Psi_j(t) \in L_0, \quad \tilde{\eta} = \int_{\tilde{\Delta}} \psi(t) d\Psi_j \in \tilde{H}.$$

It is clear that the elements $z_j = \int_{\tilde{\Delta}} d\Psi_j(t)$, $j = 1, \dots, m$, with the same structural type $d\tilde{G}(t)$ form a maximal system in $\tilde{H}$, the subspace $\tilde{L} = \bigoplus_{j=1}^m H(z_j)$ coinciding with the space $\tilde{H}$. Thus, instead of the primary systems $y_1, \dots, y_m$ and $y'_1, \dots, y'_n$ with the structural type $dG(t)$ we may consider the systems of elements z_j , $j = 1, \dots, m$, and

$$z'_k = \int_{\tilde{\Delta}} d\Psi'_k(t), \quad k = 1, \dots, n \quad (\Psi'_k(t) = P_t y'_k, \quad t_0 < t < T)$$

in the space $\tilde{H}$ with the structural type $d\tilde{G}(t)$. As established above (in the case $\tilde{L} = \tilde{H}$), we have the inequality $n \leq m$.

Let us return to the discussion of the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, with the ordered structural functions $F_j(t) = E|X_j(t)|^2$, $t_0 < t < T$ [see (2.4)].

Let $Y(t) = \{Y_k(t)\}_1^N$, $t_0 < t < T$, be a process with uncorrelated components of the form $Y_k(t) = P_t y_k$, $t_0 < t < T$, whose structural functions $G_k(t) = E|Y_k(t)|^2$, $t_0 < t < T$, are also ordered:

$$dG_1(t) > dG_2 > \dots > dG_N(t).$$

Lemma 2. *The following relations hold:*

$$\begin{aligned} N &\leq M, \\ dG_k(t) &< dF_k(t), \quad k = 1, \dots, N. \end{aligned} \tag{2.7}$$

PROOF. Suppose that some measure $dG_n(t)$ is not subordinate to the corresponding measure $dF_n(t)$. Then there exists a non-zero

measure $d\sigma(t)$, subordinate to $dG_n(t)$ and orthogonal to $dF_n(t)$. Since the measures $dF_j(t)$ with $j \geq n$ are subordinate to $dF_n(t)$, they will be orthogonal to $d\sigma(t)$. On the other hand, $d\sigma(t)$ is subordinate to a maximal structural type $dF_1(t)$ and, generally, some $dF_j(t)$, $1 \leq j \leq m$, where the number m of all such structural types from the sequence $dF_1(t) > dF_2(t) > \dots$ is strictly less than n . If $x_1, x_2, \dots, x_M$ are elements generating an innovation process $X(t) = \{X_j(t)\}_1^M$ ($X_j(t) = P_t x_j$, $t_0 < t < T$), then, taking into account the structure of the orthogonal subspaces $H(x_k)$ [see (2.1), (2.2)], it is easy to see that the elements $z_j = \int_{\Delta} dX_j(t)$, $j = 1,$

$\dots, m$, where Δ is the support of $d\sigma(t)$, form a maximal system in $H(\xi)$ with the spectral type $d\sigma(t)$. The corresponding elements $z'_k = \int_{\Delta} dY_k(t)$, $k = 1, \dots, n$, also form a system with the spec-

tral type $d\sigma(t)$, with $n > m$, which contradicts inequality (2.6) of lemma 1. Consequently, the measure $dG_n(t)$ is subordinate to $dF_n(t)$ for any n . We have also $N \leq M$.

We shall note here the fact that if the process $Y(t) = \{Y_k(t)\}_1^N$, $t_0 < t < T$, considered is itself an innovation process: $H_t(Y) = H_t(\xi)$, $t_0 < t < T$, the processes $Y(t)$ and $X(t)$ involved in lemma 2, can be interchanged, and in this case, instead of "inequalities" (2.7) we shall have $N = M$, $dG_j(t) \sim dF_j(t)$, $j = 1, \dots, N$ [see the above given relations (2.5)].

Example. Let $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$, be an arbitrary process with uncorrelated increments. Suppose that

$$F_{ij}(t) = \mathbf{E} \xi_i(t) \overline{\xi_j(t)}, \quad i, j = 1, \dots, m.$$

Let us determine the corresponding structural types $dF_1(t) > \dots > dF_M(t)$. Clearly

$$dF^*(t) = \sum_{k=1}^m \sigma_k^2 dF_{kk}(t),$$

will be a maximal type where strictly positive numbers $\sigma_1^2, \dots, \sigma_m^2$

are chosen so that the function $\sum_k \sigma_k^2 F_{kk}(t)$ is bounded. Instead of the structural types $dF_j(t)$ one can indicate the densities $f_i(t) = dF_j(t)/dF^*(t)$ or the sets

$$\Delta_j = \{t; f_j(t) > 0\},$$

that is, the supports of the measures $dF_j(t)$, $j = 1, \dots, M$. In our case the supports Δ_j can be described taking advantage of the matrix function $f(t) = \{f_{ij}(t)\}$, $t_0 < t < T$, with the components

$$f_{ij}(t) = \sigma_i \frac{dF_{ij}(t)}{dF^*(t)} \sigma_j, \quad i, j = 1, \dots, m.$$

Namely, if $\lambda_1(t) \geq \lambda_2(t) \geq \dots \geq \lambda_M(t)$ is the ordered system of all non-zero eigenvalues of the matrix function $f(t)$, then

$$\Delta_j = \{t; \lambda_j(t) > 0\}, \quad j = 1, \dots, M.$$

The corresponding innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, with ordered structural types can be obtained as

$$X_j(t) = \int_{t_0}^t \sum_{i=1}^m \sigma_i^2 u_{ij}(s) d\xi_i(s), \quad j = 1, \dots, M,$$

where $u(t) = \{u_{ij}(t)\}$ is the unitary transformation bringing the positive matrix $f(t) = \{f_{ij}(t)\}$ to a diagonal form (with the elements $\lambda_1(t) \geq \dots \geq \lambda_M(t)$, $\lambda_k(t) \equiv 0$ for $M < k \leq n$, on the main diagonal).

2. Subordinate processes. Consider a pair of processes: $\xi(t) = \{\xi_i(t)\}_1^m$ and $\eta(t) = \{\eta_k(t)\}_1^n$. We say that the process $\eta(t)$ can be obtained from the process $\xi(t)$ by means of a non-anticipatory transformation, if

$$H_t(\eta) \subseteq H_t(\xi), \quad t_0 < t < T. \quad (2.8)$$

Denote by $H_t(\eta)^\perp$ the orthogonal complement of the subspace $H_t(\eta)$ in the space $H(\eta)$; $H_t(\eta)^\perp = H(\eta) \ominus H_t(\eta)$. Similarly,

$H_t(\xi)^\perp = H(\xi) \ominus H_t(\xi)$. We shall say that the process $\eta(t)$ is *subordinate* to the process $\xi(t)$, if

$$H_t(\eta) \subseteq H_t(\xi) \text{ and } H_t(\eta)^\perp \subseteq H_t(\xi)^\perp, \quad t_0 < t < T. \quad (2.9)$$

The second of the relations (2.9) implies that any variable $y \in H(\eta)$, orthogonal to the subspace $H_t(\eta)$, must be also orthogonal to a larger subspace $H_t(\xi)$.

Let us look at two main examples involving subordinate processes.

Example (canonical representation). Let $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$ be a process with uncorrelated increments and let:

$$\eta_k(t) = \int_{t_0}^t \sum_{j=1}^m c_{kj}(t, s) d\xi_j(s), \quad t_0 < t < T; \quad (2.10)$$

$$k = 1, \dots, n.$$

Denote by Q_t the operators of orthogonal projection onto the subspaces $H_t(\eta)$, $t_0 < t < T$. The expression (2.10) is called a *canonical representation*, if the projections of the variables $\eta_k(t)$ onto the subspaces $H_u(\eta)$, $u \leq t$, are obtained by the formulas

$$Q_u \eta_k(t) = \int_{t_0}^u \sum_{j=1}^m c_{kj}(t, s) d\xi_j(s), \quad t_0 < t < T, \quad (2.11)$$

$$k = 1, \dots, n.$$

(For example, this holds in the case where $\xi(t)$, $t_0 < t < T$, is an innovation process for $\eta(t)$, $t_0 < t < T$.)

Obviously, representing the process $\eta(t) = \{\eta_k(t)\}_1^n$ as the stochastic integral (2.10) is feasible if and only if the relation (2.8) is satisfied. Under the condition (2.11) we have

$$\eta_k(t) - Q_u \eta_k(t) = \int_u^t \sum_{j=1}^m c_{kj}(t, s) d\xi_j(s);$$

it is seen that the variables $\eta_k(t) - Q_u \eta_k(t)$, $t \geq u$; $k = 1, \dots, n$, generating the subspace $H_u(\eta)^\perp$, are orthogonal to $H_u(\xi)$, and, therefore, the relation (2.9) holds, i.e., the process $\eta(t) = \{\eta_k(t)\}_1^n$, $t_0 < t < T$, is subordinate to the process with uncorrelated increments $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$.

Example. Let $\xi(t)$, $t_0 < t < T$, be a random process with the stochastic differential

$$d\xi(t) = a(t) dt + dW(t),$$

more precisely,

$$\xi(t) = \int_{t_0}^t a(s) ds + W(t), \quad t_0 < t < T, \quad (2.12)$$

where $a(t)$, $t_0 < t < T$, is some random function with mean $\mathbf{E}a(t) = 0$ and finite second moment $\mathbf{E}|a(t)|^2$,

$$\int_{t_0}^t \mathbf{E}|a(s)|^2 ds < \infty, \quad t_0 < t < T,$$

and $W(t)$, $t_0 < t < T$, is a process with uncorrelated increments, such that with $h > 0$

$$W(t+h) - W(t) \perp H_t(\xi), \quad t_0 < t < T$$

(the variables $W(t+h) - W(t)$, $h > 0$ are uncorrelated with $\xi(s)$, $s \leq t$).

Consider the process $\eta(t)$, $t_0 < t < T$, resulting from the process $\xi(t)$, $t_0 < t < T$, by means of a linear non-anticipatory transformation of the form

$$\begin{aligned} \eta(t) &= \xi(t) - \int_{t_0}^t \hat{a}(s) ds = \\ &= \int_{t_0}^t [a(s) - \hat{a}(s)] ds + W(t), \quad t_0 < t < T, \end{aligned}$$

where $\hat{a}(s) = P_s a(s)$ designates the projection of the variable $a(s)$ onto the subspace $H_s(\xi)$. Since $a(s) - \hat{a}(s) \perp H_t(\xi)$ for $s \geq t$, the increments

$$\eta(t+h) - \eta(t) = \int_t^{t+h} [a(s) - \hat{a}(s)] ds + [W(t+h) - W(t)]$$

possess the same property as that of the increments $W(t+h) - W(t)$, namely, with $h > 0$

$$\eta(t+h) - \eta(t) \perp H_t(\xi), \quad t_0 < t < T.$$

and

$$H_t(\eta) \subseteq H_t(\xi), \quad t_0 < t < T,$$

so that $\eta(t), t_0 < t < T$, is a process with uncorrelated increments. For this type of process the subspaces $H_t(\eta)^\perp$ are generated by the increments $\eta(t+h) - \eta(t), h > 0$, and we have seen that they are orthogonal to the corresponding spaces $H_t(\xi)$, and, therefore,

$$H_t(\eta)^\perp \subseteq H_t(\xi)^\perp, \quad t_0 < t < T,$$

i.e., the process $\eta(t)$ is subordinate to the process $\xi(t), t_0 < t < T$.

Let us now consider an arbitrary pair of processes $\xi(t) = \{\xi_i(t)\}_1^m$ and $\eta(t) = \{\eta_k(t)\}_1^n, t_0 < t < T$. Let $dF_1(t) > \dots > dF_M(t)$ and $dG_1(t) > \dots > dG_N(t)$ be the ordered structural types of their innovation processes.

Theorem 1. *If the process $\eta(t), t_0 < t < T$, is subordinate to the process $\xi(t), t_0 < t < T$, then*

$$\begin{aligned} N &\leq M, \\ dG_k(t) &\leq dF_k(t), \quad k = 1, \dots, N. \end{aligned} \tag{2.13}$$

PROOF. Denote by P_t and Q the operators of the orthogonal projection onto the subspaces $H_t(\xi)$ and $H_t(\eta), t_0 < t < T$. Due to

(2.9) we have

$$P_t H_t(\eta) = H_t(\eta), \quad P_t H_t(\eta)^\perp = 0$$

and

$$\begin{aligned} P_t H(\eta) &= P_t [H_t(\eta) \oplus H_t(\eta)^\perp] = P_t H_t(\eta) = \\ &= H_t(\eta) = Q_t H(\eta). \end{aligned} \quad (2.14)$$

For the projectors P_t and Q_t it follows that the operator P_t on the invariant subspace $H(\eta) \subseteq H(\xi)$ coincides with the operator Q_t , $t_0 < t < T$. Consequently, the innovation process $Y(t) = \{Y_k(t)\}_1^N$ for the process $\eta(t)$, $t_0 < t < T$, is obtained by the formula

$$Y_k(t) = P_t y_k, \quad k = 1, \dots, N,$$

where $y_1, \dots, y_N$ are variables in the subspace $H(\eta) \subseteq H(\xi)$, and Theorem 1 is therefore just a corollary of lemma 2 proved before.

We shall note here that the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$, for the process $\xi(t)$, $t_0 < t < T$, has the following property. The integral representation

$$\eta_k(t) = \int_{t_0}^t \sum_{j=1}^M c_{kj}(t, s) dX_j(s), \quad t_0 < t < T, \\ k = 1, \dots, n,$$

of the subordinate process $\eta(t) = \{\eta_k(t)\}_1^n$, $t_0 < t < T$, is canonical:

$$Q_u \eta_k(t) = P_u \eta_k(t) = \int_{t_0}^u \sum_{j=1}^M c_{kj}(t, s) dX_j(s), \quad u \leq t.$$

When considering the process $\eta(t)$, $t_0 < t < T$, obtained from the process $\xi(t)$, $t_0 < t < T$ by means of some non-anticipatory linear transformation ($H_t(\eta) \subseteq H_t(\xi)$, $t_0 < t < T$) a question may

arise whether this transformation is "reversible," more precisely, a question whether the condition

$$H_t(\xi) = H_t(\eta), \quad t_0 < t < T. \quad (2.15)$$

is satisfied.

If the transformation considered is such that the process $\eta(t)$ is subordinate to the process $\xi(t)$, $t_0 < t < T$, then the condition (2.15) is equivalent to the (generally speaking, weaker) condition

$$H(\xi) = H(\eta), \quad (2.16)$$

since in this case

$$H_t(\xi) = P_t H(\xi) = Q_t H(\eta) = H_t(\eta)$$

with all $t, t_0 < t < T$ [see (2.14)].

Theorem 2. *If the process $\eta(t)$ is subordinate to the process $\xi(t)$, $t_0 < t < T$, and they have innovation processes of the same multiplicity*

$$M < \infty \\ (dG_k(t) \sim dF_k(t), \quad k = 1, \dots, M),$$

then the relation (2.15) holds.

Before we prove this theorem, we shall note that it is false in the case of infinite multiplicity $M = \infty$. For instance, if $\xi(t) = \{\xi_i(t)\}_1^\infty$, $t_0 < t < T$ is a process with orthogonal components, each of which is a process with orthogonal increments, the structural function $F(t) = \mathbf{E} |\xi_i(t)|^2$, $t_0 < t < T$, being the same for all $i = 1, 2, \dots$, then the process $\eta(t) = \{\eta_k(t)\}_1^\infty$, $t_0 < t < T$, with the components $\eta_k(t) = \xi_{k+1}(t)$, $k = 1, 2, \dots$, will be of the same type as that of the process $\xi(t) = \{\xi_i(t)\}_1^\infty$, and will be subordinate to $\xi(t)$, $t_0 < t < T$, but, obviously, $H(\eta)$ does not contain the values $\xi_1(t)$, $t_0 < t < T$, such that $H(\eta) \neq H(\xi)$.

PROOF OF THEOREM 2. Suppose that $H(\eta) \neq H(\xi)$ and take the non-zero element $x \in H(\xi) \ominus H(\eta)$. Let $F(t) = \mathbf{E} |P_t x|^2$,

$t_0 < t < T$. From among the ordered structural types $dF_1(t) \succ \dots \succ dF_M(t)$ of the process $\xi(t)$, $t_0 < t < T$, distinguish those which the type of $dF(t)$ is subordinate to; let these be $dF_1(t) \succ \dots \succ dF_p(t)$ where $dF_p(t)$ is the most "junior" of them ($p \leq M < \infty$). If $x_1, \dots, x_M$ is a system of elements in the space $H(\xi)$ generating the innovation process $X(t) = \{X_j(t)\}_1^M$, $t_0 < t < T$; $X_j(t) = P_t x_j$, and $F_j(t) = \mathbf{E} | P_t x_j |^2$, $t_0 < t < T$; $j = 1, \dots, M$, then the system of elements

$$\tilde{x}_j = \int_{\Delta} dX_j(t), \quad j = 1, \dots, p,$$

where Δ is the support of the measure $dF(t)$, will be, obviously, the maximal orthogonal system with the structural type $dF(t)$ (see the definition on page 15).

Without loss of generality, one can assume that $dG_j(t) = dF_j(t)$ having taken the corresponding elements $y_1, \dots, y_M \in H(\eta)$ generating the innovation process $Y(t) = \{Y_j(t)\}_1^M$, $t_0 < t < T$. ($Y_j(t) = P_t y_j$, $G_j(t) = \mathbf{E} | P_t y_j |^2$, $t_0 < t < T$). The orthogonal system of elements

$$\tilde{y}_j = \int_{\Delta} dY_j(t), \quad j = 1, \dots, p,$$

with the same structural type $dF(t)$ and the multiplicity p as those of the maximal system $\tilde{x}_j$, $j = 1, \dots, p$, can be complemented by a preselected element x , and, therefore, in $H(\xi)$ there can be found a system $\tilde{y}_1, \dots, \tilde{y}_p, x$ of multiplicity $p + 1$, larger than the multiplicity p which is not feasible [see Inequality (2.6)]. Consequently, the assumption that $H(\eta) \neq H(\xi)$ is false, and, therefore, the theorem is proved.

§3. Some Examples Involving Innovation Processes

1. Markov processes. We shall consider below two major classes of processes—Markov and wide-sense stationary processes.

A multidimensional random process $\xi(t) = \{\xi_i(t)\}_1^m$ is called a wide-sense Markov process if for any t the projections $P_t \xi_i(t+h)$ of the variables $\xi_i(t+h)$, $h \geq 0$ onto "the past" of $H_t(\xi)$ belong to the closed linear hull of the values $\xi_i(t)$, $i = 1, \dots, m$.

We restrict ourselves to considering only finite-dimensional processes. In this case for any $s \leq t$

$$P_s \xi_i(t) = \sum_{j=1}^m c_{ij}(t, s) \xi_j(s), \quad i = 1, \dots, m, \quad (3.1)$$

where the coefficients $c_{ij}(t, s)$ are easily defined by the correlation functions

$$B_{ij}(t, s) = \mathbf{E} \xi_i(t) \overline{\xi_j(s)}, \quad i, j = 1, \dots, m.$$

For example, in the case of linearly independent values $\xi_j(s)$, $j = 1, \dots, m$, when the matrix $B(s, s) = \{B_{ij}(s, s)\}$ is non-singular with inverse $B(s, s)^{-1} = \{b_{kj}(s, s)\}$, the coefficients in the expression (3.1) can be found by the formula

$$c_{ij}(t, s) = \sum_k B_{ik}(t, s) b_{kj}(s, s), \quad i, j = 1, \dots, m.$$

According to the general assumption (1.1)

$$\lim_{s \rightarrow t} P_s \xi_i(t) = \xi_i(t), \quad i = 1, \dots, m,$$

and in the case of linearly independent values $\xi_i(t)$, $i = 1, \dots, m$, there is a certain interval $t_1 \leq s \leq t$, such that the variables

$$\eta_i(s) = P_s \xi_i(t), \quad i = 1, \dots, m,$$

form the basis in the linear hull of the values $\xi_i(s)$, $i = 1, \dots, m$. Hence the orthogonal complement $H_t(\xi) \ominus H_{t_1}(\xi)$ to the subspace $H_{t_1}(\xi)$ in the space $H_t(\xi)$ generated by all the variables

$$\xi_i(s) - P_{t_1} \xi_i(s), \quad t_1 \leq s \leq t; \quad i = 1, \dots, m,$$

coincides with the closed linear hull of the variables $\eta_i(s) - \eta_i(t_1) = \eta_i(s) - P_{t_1} \eta_i(s)$, $t_1 \leq s \leq t$; $i = 1, \dots, m$.

The random process $\eta(s) = \{\eta_i(s)\}_1^m$, $t_1 \leq s \leq t$, is a process with uncorrelated increments and possesses the property that

$$H_{s_2}(\xi) \ominus H_{s_1}(\xi) = H_{s_2}(\eta) \ominus H_{s_1}(\eta) \quad (3.2)$$

for any $t_1 \leq s_1 \leq s_2 \leq t$, and, in this sense, is an innovation process for $\xi(s) = \{\xi_i(s)\}_1^m$, $t_1 \leq s \leq t$. (One can then go over to an innovation process with uncorrelated components and ordered structural types as in the example on page 18.)

The simplest example of a wide-sense Markov process is the multidimensional process $\xi(t) = \{\xi_i(t)\}_1^m$, $t \geq t_0$, satisfying the system of stochastic differential equations of the form

$$d\xi_i(t) = \sum_{j=1}^m A_{ij}(t) \xi_j(t) dt + d\eta_i(t), \quad i = 1, \dots, m, \quad (3.3)$$

or, in a matrix form,

$$d\xi(t) = A(t) \xi(t) dt + d\eta(t),$$

where $A(t) = \{A_{ij}(t)\}$ is a continuous matrix function (non-random), and $\eta(t) = \{\eta_i(t)\}_1^m$, $t \geq t_0$ is a process with uncorrelated increments.

A solution of the system (3.3) with the prescribed initial values $\xi(t_0) = \{\xi_i(t_0)\}_1^m$, more precisely, a solution of the integral equation

$$\xi(t) = \xi(t_0) + \int_{t_0}^t A(s) \xi(s) ds + [\eta(t) - \eta(t_0)], \quad t \geq t_0. \quad (3.3')$$

can be represented as

$$\xi(t) = R(t, t_0) \xi(t_0) + \int_{t_0}^t R(t, s) d\eta(s), \quad (3.4)$$

where $m \times m$ matrix function $R(t, s) = \{R_{ij}(t, s)\}$, $t \geq s$ —the so-called “resolvent”—is a solution of the ordinary differential equation

$$\frac{dR(t, s)}{dt} = A(t) R(t, s), \quad t > s,$$

with the initial condition $R(s, s) = I$ (I is the unit matrix). Indeed, by definition we have

$$R(t, s) = I + \int_s^t A(u) R(u, s) du$$

and

$$\begin{aligned} \int_{t_0}^t R(t, s) d\eta(s) &= \int_{t_0}^t \left[I + \int_s^t A(u) R(u, s) du \right] d\eta(s) = \\ &= [\eta(t) - \eta(t_0)] + \int_{t_0}^t \left[\int_s^t A(u) R(u, s) du \right] d\eta(s) = \\ &= [\eta(t) - \eta(t_0)] + \int_{t_0}^t A(u) \left[\int_{t_0}^u R(u, s) d\eta(s) \right] du. \end{aligned}$$

It is seen that

$$\tilde{\xi}(t) = \int_{t_0}^t R(t, s) d\eta(s), \quad t \geq t_0,$$

is a solution of the integral equation (3.3') with the initial value $\tilde{\xi}(t_0) = 0$, the difference $x(t) = \xi(t) - \tilde{\xi}(t)$, $t \geq t_0$, as the solution of the homogeneous differential equation $dx(t)/dt = A(t)x(t)$ with the initial condition $x(t_0) = \xi(t_0)$, has the form

$$x(t) = R(t, t_0) \xi(t_0).$$

Assume that the increments $\eta_i(t) - \eta_i(s)$, $s, t \geq t_0$; $i = 1, \dots, m$, are uncorrelated with the initial values $\xi_j(t_0)$, $j = 1, \dots, m$. Then, if we redistribute the process $\eta(t) = \{\eta_i(t)\}_1^m$, $t < t_0$, more precisely, if, instead of the initial process $t < t_0$, $\eta(t)$, we consider the process with uncorrelated increments $\eta(t) - \eta(t_0) + \xi(t_0)$, one can (without change of the notation) consider

$$\eta_i(t_0) = \xi_i(t_0), \quad i = 1, \dots, m \quad (3.5)$$

and rewrite the expression (3.4) as

$$\xi(t) = \int_{t_0}^t R(t, s) d\eta(s). \quad (3.6)$$

Obviously, the subspaces $H_t(\eta)$ are generated by the values $\xi_i(t_0)$ and the increments $\eta_i(s) - \eta_i(t_0)$, $t_0 < s \leq t$. Using the fact that the resolvent $R(t, s)$ satisfies the condition

$$R(t+h, s) = R(t+h, t) \cdot R(t, s)$$

for all $t \geq s$ and $h \geq 0$, we obtain from the formula (3.6) that

$$\begin{aligned} R(t+h, t) \xi(t) &= \int_{t_0}^t R(t+h, t) R(t, s) d\eta(s) = \\ &= \int_{t_0}^t R(t+h, s) d\eta(s) \end{aligned}$$

and

$$\xi(t+h) - R(t+h, t) \xi(t) = \int_t^{t+h} R(t+h, s) d\eta(s).$$

It is seen that the differences

$$\xi_i(t+h) - \sum_{j=1}^m R_{ij}(t+h, t) \xi_j(t), \quad i = 1, \dots, m,$$

are orthogonal to the subspace $H_t(\xi)$, i.e., the variables

$$\sum_{j=1}^m R_{ij}(t+h, t) \xi_j(t) = \int_{t_0}^t \sum_{j=1}^m R_{ij}(t+h, s) d\eta_j(s)$$

are the projections of the values $\xi_i(t+h)$, $i = 1, \dots, m$, onto the subspaces $H_t(\xi) \subseteq H_t(\eta)$.

Thus, the process $\xi(t) = \{\xi_i(t)\}_1^m$, $t \geq t_0$, is a wide-sense Markov process, and the formula (3.6) yields its canonical representation by means of the process $\eta(t) = \{\eta_i(t)\}_1^m$, $t \geq t_0$, with uncorrelated increments, described above [see (3.5)].

Obviously, $H_t(\eta) \subseteq H_t(\xi)$ [see (3.3')], i.e., this process is an innovation process for $\xi(t) = \{\xi_i(t)\}_1^m$, $t \geq t_0$; in fact

$$H_t(\xi) = H_t(\eta)$$

for all $t \geq t_0$.

2. Stationary processes. Consider a wide-sense stationary process $\xi(t) = \{\xi_k(t)\}_1^m$, $-\infty < t < \infty$, i.e., such that the correlation functions $B_{kj}(t, s) = E \xi_k(t) \xi_j(s)$ depend only on the difference $t - s$:

$$B_{kj}(t, s) = B_{kj}(t - s), \quad k, j = 1, \dots, m.$$

A stationary process $\xi(t) = \{\xi_i(t)\}_1^n$, $-\infty < t < \infty$, is said to be regular, if

$$\bigcap_t H_t(\xi) = 0. \quad (3.7)$$

As is well known,* a regular process has a spectral density $f(\lambda) = \{f_{kj}(\lambda)\}$; this is a positive-definite matrix function of λ ,

*See for instance, Yu. A. Rozanov, *Statsionarnyye sluchaynyye protsessy* (Stationary random processes), Fizmatgiz, Moscow, 1963.

$-\infty < \lambda < \infty$, whose components $f_{kj}(\lambda)$ are related to correlation functions by means of the relation

$$B_{kj}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f_{kj}(\lambda) d\lambda, \quad k, j = 1, \dots, m.$$

Furthermore, the condition (3.7) is equivalent to the assumption that the spectral density $f(\lambda)$ has almost everywhere the same rank $n \leq m$ and is representable as

$$f(\lambda) = \frac{1}{2\pi} \varphi(\lambda) \cdot \varphi(\lambda)^* \quad \text{a.e.}, \quad (3.8)$$

where the $m \times n$ -matrix function $\varphi(\lambda) = \{\varphi_{kj}(\lambda)\}$ satisfies the conditions

$$\int_{-\infty}^{\infty} \|\varphi(\lambda)\|^2 d\lambda < \infty, \quad (3.9)$$

$$\int_{-\infty}^{\infty} e^{i\lambda t} \varphi(\lambda) d\lambda = 0 \quad \text{for } t < 0,$$

and $\varphi(\lambda)^*$ designates the matrix conjugated to $\varphi(\lambda)$.

Denote by H^2 the class of all matrix functions $\varphi(\lambda)$ satisfying the conditions (3.9). The Fourier transform

$$c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(\lambda) d\lambda \quad (3.10)$$

of the functions in H^2 vanishes for $t < 0$, and

$$\varphi(\lambda) = \int_0^{\infty} e^{-i\lambda t} c(t) dt$$

continues analytically into the lower semiplane $\text{Im } \lambda < 0$ of the complex variable λ .

Starting from the spectral representation

$$\xi_k(t) = \int_{-\infty}^{\infty} e^{i\lambda t} d\Phi_k(\lambda), \quad k = 1, \dots, m, \quad (3.11)$$

where $\Phi(\lambda) = \{\Phi_k(\lambda)\}_1^m$, $-\infty < \lambda < \infty$, is a process with uncorrelated increments, such that

$$\mathbf{E} \Phi_k(\mu) \overline{\Phi_j(\mu)} = \int_{-\infty}^{\mu} f_{kj}(\lambda) d\lambda, \quad k, j = 1, \dots, m,$$

the following expression can be obtained for the stationary process in question:

$$\xi_k(t) = \int_{-\infty}^t \sum_{j=1}^n c_{kj}(t-s) d\eta_j(s), \quad k = 1, \dots, m, \quad (3.12)$$

where the $m \times n$ matrix function $c(t) = \{c_{kj}(t)\}$, $t \geq 0$, is the Fourier transform of the matrix function $\varphi(\lambda) = \{\varphi_{kj}(\lambda)\}$ of the class $\mathbf{H}^2$ satisfying the condition (3.8), and the stochastic integral itself on the right-hand side of (3.12) corresponds to the uncorrelated increments

$$\eta_j(t) - \eta_j(s) = \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - e^{i\lambda s}}{i\lambda} \varphi(\lambda)^{-1} d\Phi(\lambda), \quad (3.13)$$

$$j = 1, \dots, n.$$

Let us make it clear here that, first, $\varphi(\lambda)^{-1}$ is a $n \times m$ matrix function, the inverse of $\varphi(\lambda)$ (i.e., the product $\varphi(\lambda)^{-1} \varphi(\lambda)$ is an $n \times n$ unit matrix), and, second, on non-overlapping intervals (s_1, t_1) and (s_2, t_2) the variables $\eta_j(t_1) - \eta_j(s_1)$ and $\eta_j(t_2) - \eta_j(s_2)$ are uncorrelated, with

$$\mathbf{E}|\eta_j(t) - \eta_j(s)|^2 = t - s, \quad j = 1, \dots, n, \quad (3.14)$$

and the variables $\eta_k(t_1) - \eta_k(s_1)$ and $\eta_j(t_2) - \eta_j(s_2)$ are uncorrelated for any integrals, if $k \neq j$. The expression (3.12) can be regarded as a non-anticipatory linear transformation of the n -dimensional generalized process $\dot{\eta}(t) = \{\dot{\eta}_j(t)\}_1^n$ of the "white noise" type, rewriting (3.12) as

$$\xi(t) = \int_{-\infty}^t c(t-s) \dot{\eta}(s) ds. \quad (3.15)$$

Denote by $H_t(\eta)$ the subspace generated by all variables $\eta_j(t) - \eta_j(s)$, $s \leq t$, $j = 1, \dots, n$, and by Q_t the operator projecting onto $H_t(\eta)$.

Since

$$H_t(\eta) \supseteq H_t(\xi), \quad -\infty < t < \infty,$$

the following inequalities hold for any $h > 0$:

$$\begin{aligned} \int_0^h \sum_{k=1}^m \sum_{j=1}^n |c_{kj}(s)|^2 ds &= \sum_{k=1}^m \mathbf{E}|\xi_k(t+h) - Q_t \xi_k(t+h)|^2 \leq \\ &\leq \sum_{k=1}^m \mathbf{E}|\xi_k(t+h) - P_t \xi_k(t+h)|^2, \end{aligned}$$

where P_t , as before, denotes the operator of projecting onto the subspace $H_t(\xi)$. If we assume that among the matrix functions $\varphi(\lambda)$ of class $\mathbf{H}^2$ satisfying the condition (3.8) there would be a function $\varphi^0(\lambda)$ for which the appropriate increments $\eta_j^0(t) - \eta_j^0(s)$, $j = 1, \dots, n$, yield equality

$$H_t(\eta^0) = H_t(\xi), \quad (3.16)$$

then the matrix function $c^0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda t} \varphi^0(\lambda) d\lambda$ with respect

to all the matrix functions $c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda t} \varphi(\lambda) d\lambda$ will be maximal in the sense that

$$\begin{aligned} \int_0^h \sum_{k=1}^m \sum_{j=1}^n |c_{kj}^0(s)|^2 ds &= \sum_{k=1}^m \mathbb{E} |\xi_k(t+h) - P_t \xi_k(t+h)|^2 \geq \\ &\geq \int_0^h \sum_{k=1}^m \sum_{j=1}^n |c_{kj}(s)|^2 ds \end{aligned}$$

for all $h \geq 0$. Such function $\varphi^0(\lambda) \in \mathbf{H}^2$ exists, indeed, and is defined by the condition that it is maximal in the true meaning of this word:

$$\|\varphi^0(\lambda)\|^2 = \sum_{k=1}^n \sum_{j=1}^m |\varphi_{kj}^0(\lambda)|^2 \geq \sum_{k=1}^n \sum_{j=1}^m |\varphi_{kj}(\lambda)|^2 = \|\varphi(\lambda)\|^2 \quad (3.17)$$

for all complex λ , $\text{Im } \lambda < 0$.

We have actually described an innovation process given the appropriate increments $\eta_j^0(t) - \eta_j^0(s)$, $j = 1, \dots, m$. As the general formula (3.14) indicates, the innovation process multiplicity M is equal to n —the rank of the spectral density $f(\lambda)$:

$$M = n,$$

and in the ordered system of the structural types $dF_1(t) > \dots > dF_M(t)$ [see (2.4)] all $dF_j(t)$ are equivalent to Lebesgue measure:

$$dF_j(t) \sim dt, \quad j = 1, \dots, M. \quad (3.18)$$

It will be different if the stationary process $\xi(t) = \{\xi_i(t)\}_1^m$, $-\infty < t < \infty$, is considered only on a finite interval $t_0 < t < T$ (or with $t > t_0$).

Let us make this clear using an example involving a one-dimensional stationary process $\xi(t)$ with a spectral density of the form

$$f(\lambda) = \frac{1}{2\pi |P(i\lambda)|^2}, \quad (3.19)$$

where $P(z) = \sum_{k=0}^n p_k z^k$ is the polynomial of the power n , whose roots lie entirely in the left-hand half-plane $\operatorname{Re} z < 0$ (in particular, $p_0 \neq 0$). In this case

$$\varphi(\lambda) = \frac{1}{P(i\lambda)},$$

will be a maximal function satisfying the condition (3.8), and the formula (3.13) yields

$$\begin{aligned} \eta(t) - \eta(s) &= \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - e^{i\lambda s}}{i\lambda} P(i\lambda) d\Phi(\lambda) = \\ &= \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - e^{i\lambda s}}{i\lambda} p_0 d\Phi(\lambda) + \\ &+ \int_{-\infty}^{\infty} (e^{i\lambda t} - e^{i\lambda s}) \sum_{k=0}^{n-1} p_{k+1} (i\lambda)^k d\Phi(\lambda) = \\ &= p_0 \int_s^t \xi(u) du + \sum_{k=0}^{n-1} p_{k+1} [\xi^{(k)}(t) - \xi^{(k)}(s)]. \end{aligned} \quad (3.20)$$

If we rewrite this relation in the differential form

$$d\eta(t) = \sum_{k=0}^{n-1} p_k \xi^{(k)}(t) dt + p_n d\xi^{(n-1)}(t),$$

then it can be readily seen that the multidimensional process

$\{\xi_k(t)\}_1^n, t \geq t_0$, with the components

$$\xi_k(t) = \xi^{(k-1)}(t), \quad k = 1, \dots, n,$$

satisfies the system of stochastic differential equations of the type (3.3):

$$\begin{aligned} d\xi_k(t) &= \xi_{k+1}(t) dt, \quad k = 1, \dots, n-1, \\ d\xi_n(t) &= \left(\frac{1}{p_n} \sum_{k=1}^n \xi_k(t) \right) dt + \frac{1}{p_n} d\eta(t). \end{aligned} \quad (3.21)$$

Furthermore, as in the case of general stationary processes, the variables $\eta(t) - \eta(t_0), t \geq t_0$, defined by the expression (3.20), are orthogonal to the closed linear hull of all the variables $\xi(s), s \leq t_0$, and, in particular, to

$$\xi_k(t_0) = \xi^{(k-1)}(t_0), \quad k = 1, \dots, n. \quad (3.22)$$

Therefore, the multidimensional process $\{\xi_k(t)\}_1^n, t \geq t_0$, is of the type considered, and the results obtained above (see page 30) imply that in our case the process $\{\eta_k(t)\}_1^n, t \geq t_0$, with the components

$$\begin{aligned} \eta_k(t) &= \xi_k(t_0), \quad k = 1, \dots, n-1, \\ \eta_n(t) &= \eta(t) - \eta(t_0) + \xi_n(t_0). \end{aligned} \quad (3.23)$$

will be an innovation process for $\{\xi_k(t)\}_1^n, t \geq t_0$.

Clearly, this process will be an innovation process also for the initial stationary process $\xi(t), t \geq t_0$.

Chapter II

REGULAR PROCESSES

§1. Isometric Families. Some Examples

Let H_t , $t_0 < t < T$, be a family of subspaces in a Hilbert space H satisfying the conditions (1.1), Chap. I. Let U_t , $t_0 < t < T$, be another family (in a Hilbert space U). We shall call these families isometric if there exists an isometric operator X from U into H , such that

$$H_t = XU_t, \quad t_0 < t < T. \quad (1.1)$$

In order to avoid new notation, we shall consider that H and U are the closures of the unions of all subspaces H_t , $t_0 < t < T$, and U_t , $t_0 < t < T$, respectively; then X is a unitary operator mapping U onto H .

Denote by P_t the operator projecting onto H_t and by Q_t , the operator projecting onto U_t , $t_0 < t < T$. Clearly, under the condition (1.1)

$$P_t = XQ_tX^{-1}, \quad t_0 < t < T \quad (1.2)$$

since for the unitary operator X , along with the condition (1.1), the condition $H_t^\perp = XU_t^\perp$ (where $H_t^\perp = H \ominus H_t$, $U_t^\perp = U \ominus U_t$) is also satisfied, and hence, for $h \in H_t$,

$$X^{-1}h \in U_t, \quad Q_tX^{-1}h = X^{-1}h, \quad XQ_tX^{-1}h = h,$$

and, for $h \perp H_t$

$$X^{-1}h \perp U_t, \quad Q_t X^{-1}h = 0, \quad XQ_t X^{-1}h = 0.$$

Take the arbitrary elements $u_1, \dots, u_m \in U$, generating the "innovation" process $\{Q_t u_k\}_1^N$, $t_0 < t < T$, for the family U_t , $t_0 < t < T$ (i.e., the subspaces U_k generated by all the values of $Q_t u_k$, $t_0 < t < T$, are mutually orthogonal for $k = 1, \dots, N$, and $U = \bigoplus_{k=1}^N U_k$). Set $F_k(t) = \|Q_t u_k\|^2$, $t_0 < t < T$; $k = 1, \dots, N$. Obviously, the corresponding elements $x_k = Xu_k$, $k = 1, \dots, N$, generate in the space H an innovation process $\{P_t x_k\}_1^N$, $t_0 < t < T$, exactly of the same type, since

$$P_t x_k = XQ_t u_k, \quad t_0 < t < T$$

from which it is seen that the subspaces $H_k = XU_k$ (generated by the variables $P_t x_k$, $t_0 < t < T$) are mutually orthogonal, and

$$\|P_t x_k\|^2 = \|XQ_t u_k\|^2 = \|Q_t u_k\|^2 = F_k(t),$$

$$t_0 < t < T; \quad k = 1, \dots, N.$$

Therefore, for the isometric families U_t , $t_0 < t < T$, and H_t , $t_0 < t < T$, the innovation processes are of the same type.

The converse statement also holds. Indeed, considering the "cyclic" subspaces U_k and H_k with the equivalent structural types, one can choose generating elements $u_k \in U_k$ and $x_k \in H_k$ so that their structural types coincide:

$$\|Q_t u_k\|^2 = \|P_t x_k\|^2 = F_k(t), \quad k = 1, \dots, N.$$

The spaces $U = \bigoplus_{k=1}^N U_k$ and $H = \bigoplus_{k=1}^N H_k$ are unitarily isomorphic to the space $\mathbf{C}$ of all vector functions $\mathbf{c}(t) = \{c_k(t)\}_1^N$, $t_0 \leq t < T$, with components satisfying the condition

$$\int_{t_0}^T \sum_{k=1}^N |c_k(t)|^2 dF_k(t) < \infty,$$

in which the scalar product of the elements $\mathbf{c}_1(t) = \{c_{1k}(t)\}_1^N$ and $\mathbf{c}_2(t) = \{c_{2k}(t)\}_1^N$ is defined as

$$\int_{t_0}^T \sum_{k=1}^N c_{1k}(t) \overline{c_{2k}(t)} dF_k(t). \quad (1.3)$$

Under the unitary mapping $H, U \rightarrow \mathbf{C}$, to the subspaces H_t and U_t corresponds the subspace $\mathbf{C}_t$ of all functions $\mathbf{c}(s)$, $t_0 \leq t < T$, in the space $\mathbf{C}$ such that $\mathbf{c}(s) = 0$ for $s > t$ (see Sec. 1, Chap. I, page 7).

Let us assume now that there are two families of the subspaces: H_t , $t_0 < t < T$, and U_t , $t_0 < t < T$, interrelated as

$$H_t = AU_t, \quad t_0 < t < T \quad (1.4)$$

where A is a certain linear operator on the Hilbert space U into the Hilbert space H . Or let us assume that a mapping of the type (1.4) relates the (open) subspaces H_t^0 and U_t^0 :

$$H_t^0 = AU_t^0, \quad (1.4')$$

whose closures are

$$H_t = \overline{H_t^0}, \quad U_t = \overline{U_t^0}, \quad t_0 < t < T.$$

The question is for which operators A it can be asserted that the corresponding families H_t and U_t , $t_0 < t < T$, will be isometric.

In order to clarify in what way this question is related to the innovation processes theory, we shall consider a few examples.

Let $\eta(t) = \{\eta_i(t)\}_1^m$, $t_0 < t < T$, be some "standard" process for which the innovation process type is known or even the projectors Q_t onto the subspaces $H_t(\eta)$, $t_0 < t < T$ are defined.

Let $\xi(t) = \{\xi_i(t)\}_1^m$; $t_0 < t < T$, be another process which we wish to compare with the standard process $\eta(t) = \{\eta_i(t)\}_1^m$, $t_0 < t < T$. Introduce an operator A :

$$A\eta_i(t) = \xi_i(t), \quad i = 1, \dots, m, \quad t_0 < t < T, \quad (1.5)$$

and extend it linearly to the linear hull $H^0(\eta)$ of all the variables $\eta_i(t)$, $i = 1, \dots, m$, $t_0 < t < T$ (this is possible, for example, if the above variables are linearly independent). If we denote by $H_t^0(\eta)$ the linear hull of the variables $\eta_i(s)$, $i = 1, \dots, m$, $t_0 < s \leq t$, then, obviously,

$$AH_t^0(\eta) = H_t^0(\xi)$$

and

$$H_t(\eta) = \overline{H_t^0(\eta)}, \quad H_t(\xi) = \overline{H_t^0(\xi)}, \quad t_0 < t < T.$$

We can see that the question about the operator A and $H_t = H_t(\xi)$, $U_t = H_t(\eta)$ posed above implies the following: under which conditions imposed on the operator A is the innovation process for $\xi(t)$, $t_0 < t < T$, of the same type as that of the innovation process for $\eta(t)$, $t_0 < t < T$?

Note here that for processes of the same type the problem of finding an appropriate isometric operator X is crucial:

$$H_t(\xi) = XH_t(\eta), \quad t_0 < t < T$$

[see (1.1)], with the aid of which the projectors P_t onto the subspaces $H_t(\xi)$ can be defined by the formula (1.2):

$$P_t = XQ_tX^{-1}, \quad t_0 < t < T.$$

Consider a few examples showing that the structural type can change unexpectedly in passing from the family of subspaces U_t , $t_0 < t < T$, over to the family $H_t = \overline{AU_t}$, $t_0 < t < T$, where A is a bounded linear operator.

Let $\eta(t)$, $0 < t < 1$ be a standard Wiener process. As known,

$$\eta(t) = \sum_{k=0}^{\infty} \eta_k \varphi_k(t), \quad 0 < t < 1$$

where $\eta_k = \int_0^1 \eta(t) \varphi_k(t) dt$, and $\varphi_k(t) = \sin(k + 1/2) \pi t$ are the eigenfunctions of the kernel $B(s, t) = \min(s, t)$, $0 \leq s, t \leq 1$.

Let A be the operator of projecting onto the finite-dimensional subspace L generated by the variables $\eta_0, \eta_1, \dots, \eta_k$ (note that η_k , $k = 0, 1, \dots$, is the orthogonal basis in the space $U = H(\xi)$). Consider the process

$$\xi(t) = A\eta(t) = \sum_{k=0}^{n-1} \eta_k \varphi_k(t), \quad 0 < t < 1.$$

For any $t > 0$, for the appropriate choice of the points $0 < t_1 < \dots < t_n < t$ the matrix $\{\varphi_k(t_j)\}$ will be non-singular, and from the equations

$$\sum_{k=0}^{n-1} \eta_k \varphi_k(t_j) = \xi(t_j), \quad j = 1, \dots, n,$$

we find that the variables $\eta_0, \dots, \eta_{n-1}$ are linear combinations of the values $\xi(t_1), \dots, \xi(t_n)$, and belong to the subspace $H_t(\xi)$. Obviously,

$$H_t(\xi) \equiv L, \quad 0 < t < 1,$$

and the process $X(t) = \{X_k(t)\}_1^n$ with the components $X_k(t) \equiv \eta_{k-1}$, $k = 1, \dots, n$, will be an innovation process for $\xi(t)$, $0 < t < 1$. Therefore, if the primary family $U_t = H_t(\eta)$, $0 < t < 1$, had the multiplicity $N = 1$ (the process $\eta(t)$, $0 < t < 1$, with the structural type dt , itself is an innovation process), then the family

$H_t = \overline{AU_t}$, $0 < t < 1$, would have the multiplicity $M = n$, and, the measure entirely concentrated in the point $t = 0$ would be the structural type (of multiplicity M).

As this example demonstrates, even for the operator A at the continuity points of the family U_t ($U_{t+0} = U_t$) a new family $H_t = \overline{AU_t}$ may have "jumps" ($H_{t+0} \neq H_t$).

Generally, for the simplest standard family $U_t = H_t(\eta)$, $0 < t < 1$, where $\eta(t)$ is a Wiener process, one can point out a projector A , such that the family $H_t = \overline{AU_t}$, $0 < t < 1$, will have an arbitrary preselected structural type. Thus if we take a process of the type

$$\xi(t) = \int_0^t \dot{\xi}(s) ds \quad \left(\int_0^1 \mathbf{E} |\dot{\xi}(s)|^2 ds < \infty \right),$$

for which the innovation process is of the preselected structural type (the existence of such a process $\xi(t)$, $0 < t < 1$, with any preselected structural type was discussed on page 7), and take its orthogonal standard Wiener process $W(t)$, $0 < t < 1$ ($H(W) \perp H(\xi)$), setting

$$\eta(t) = \xi(t) + W(t), \quad 0 < t < 1$$

then the family $H_t(\eta)$, $0 < t < 1$, will be isometric to the family $U_t = H_t(W)$, $0 < t < 1$ (see page 21). In this case

$$\xi(t) = A\eta(t), \quad 0 < t < 1,$$

where A is the operator projecting onto the subspace $H(\xi)$.

Let us return to the discussion of the arbitrary families of the subspaces U_t and $H_t = \overline{AU_t}$, $t_0 < t < T$ (A is a bounded linear operator).

We shall call the operator A invertible if it has a bounded linear inverse A^{-1} . For operators of this type the question about isometry of the families U_t and $H_t = \overline{AU_t}$, $t_0 < t < T$, can be easily resolved in the case where U_t , $t_0 < t < T$, is a discrete chain of subspaces;

namely,

$$U_t = \bigoplus_{t_k < t} \Delta_k, \quad (1.6)$$

where t_k , $k = 1, 2, \dots$, is a finite or countable set of points of the interval $[t_0, T)$, in which

$$\Delta_k = U_{t_k+0} \ominus U_{t_k} = \lim_{h \rightarrow +0} (U_{t_k+h} \ominus U_{t_k}) \neq 0.$$

The structural types

$$dF_1(t) \succ dF_2(t) \succ \dots \succ dF_N(t)$$

of such a family are purely discrete, with jumps at the points t_k , $k = 1, 2, \dots$. Note that $F_j(t) = \|Q_t u_j\|^2$, $t_0 < t < T$, where Q_t is the operator projecting onto the subspace U_t , and $u_1, \dots, u_N$ a complete system of "cyclic" vectors in the Hilbert space U , for which the subspaces U_j generated by the elements $Q_t u_j$, $t_0 < t < T$, are orthogonal for $j = 1, \dots, N$ and $U = \bigoplus_{j=1}^N U_j$ (see page 38). In this case the cyclic vectors $u_1, \dots, u_N$ can be selected as follows:

$$u_j = \sum_k c_k u_{kj}, \quad j = 1, \dots, N,$$

where u_{kj} , $j = 1, \dots, N_k$, is an orthonormal basis in the corresponding subspace $\Delta_k = U_{t_k+0} \ominus U_{t_k}$ of dimension N_k ($N_k \leq N$), $u_{kj} = 0$ with $j > N_k$ and $\sum_k |c_k|^2 < \infty$.

Obviously, the families U_t and $H_t = AU_t$, $t_0 < t < T$, will be isometric if and only if H_t , $t_0 < t < T$, is a purely discrete chain with jumps at the same points t_k , $k = 1, 2, \dots$, with

$$\dim(H_{t_k+0} \ominus H_{t_k}) = \dim(U_{t_k+0} \ominus U_{t_k}); \quad (1.7)$$

under this condition, having chosen any unitary operator X :

$$X(U_{t_k+0} \ominus U_{t_k}) = H_{t_k+0} \ominus H_{t_k}, \quad k = 1, 2, \dots,$$

we shall have

$$H_t = XU_t, \quad t_0 < t < T.$$

Let A be a bounded invertible operator and $H_t = AU_t$, $t_0 < t < T$. Since $AU_{t+0} \subseteq AU_{t+h} = H_{t+h}$ for all $h > 0$, we have $AU_{t+0} \subseteq \bigcap_{h>0} H_{t+h} = H_{t+0}$. Taking into account that $U_t = A^{-1}H_t$, $t_0 < t < T$, we have $A^{-1}H_{t+0} \subseteq U_{t+0}$, $H_{t+0} \subseteq AU_{t+0}$ and finally we obtain

$$H_{t+0} = AU_{t+0}, \quad t_0 < t < T, \quad (1.8)$$

The equality (1.8) demonstrates that the family H_t , $t_0 < t < T$, is a discrete chain of subspaces and can have jumps only at the same points t_k as those of the family U_t , $t_0 < t < T$, since with $U_{t+0} = U_t$ we have $H_{t+0} = AU_{t+0} = AU_t = H_t$.

It should be noted here that without invertibility of the operator A , the equality (1.8) need not in general be satisfied; moreover, it may turn out that $U_{t+0} = \lim_{h \rightarrow +0} U_{t+h} = 0$, while $H_{t+0} =$

$\lim_{h \rightarrow +0} \overline{AU_{t+h}} \neq 0$, which seems rather unexpected for the bounded

operator A (compare with the example on page 40).

Let us show that the condition (1.7) can be satisfied. Indeed, the invertible operator A transfers the direct sum of the subspaces $U_{t_k+0} = U_{t_k} + \Delta_k$ into a direct sum $H_{t_k+0} = H_{t_k} + A\Delta_k$ (more precisely, the element $x = Au$ does not belong to H_{t_k}) for no $u \in \Delta_k$ from which it follows that the subspace

$$H_{t_k+0} \ominus H_{t_k} = (I - P_{t_k}) A\Delta_k$$

(P_t designates the operator of projecting on H_t) has the same dimension as that of the subspace $A\Delta_k$ which coincides for the

invertible operator A with the dimension of the subspace

$$\Delta_k = U_{t_{k+0}} \ominus U_{t_k}$$

Therefore, the following assertion holds.

*Let U_t , $t_0 < t < T$, be a discrete chain of subspaces in a Hilbert space U and A be a bounded invertible operator on U into a Hilbert space H ; then the families U_t and $H_t = AU_t$, $t_0 < t < T$, are isometric.**

§2. Some Models of Random Processes. Regularity and a Problem of Factorization

From now on it will be more convenient to consider generalized random processes.

For example, comparing an "ordinary" process $\xi(t) = \{\xi_i(t)\}_1^m$, $t_0 < t < T$, with some standard process $\eta(t) = \{\eta_i(t)\}_1^m$, $t_0 < t < T$, a generalized process (ξ, u) , $u \in U^0$, can be introduced on the pre-Hilbert space $U^0 = H^0(\eta)$, the linear hull of all variables $\eta_i(t)$, setting

$$(\xi, u) = Au, \quad u \in U^0, \quad (2.1)$$

where A is the linear operator on U^0 into $H(\xi)$ defined by the formula (1.5) (namely, $A\eta_i(t) = \xi_i(t)$, $i = 1, \dots, m$, $t_0 < t < T$).

Imagine that a certain generalized random process (ξ, u) , $u \in U^0$ is being observed defined on a certain subspace U^0 in a Hilbert space U with a scalar product (u, v) , $u, v \in U$, in which a certain family of subspaces U_t^0 , $t_0 < t < T$, is given, such that at the time t "the observer" has available all the values of (ξ, u) , $u \in U_t^0$. Assume that U^0 is dense in U : $U = U^0$, and in the Hilbert space given is a correlation operator B :

*This is true for the invertible operator A for an arbitrary family U_t , $t_0 < t < T$, as well.

$$E(\xi, u)(\xi, v) = (Bu, v), \quad u, v \in U^0.$$

Denote by H_t the closure of the subspace of all values (ξ, u) , $u \in U_t^0$. The question is under which conditions imposed on the correlation operator B the families H_t , $t_0 < t < T$, and $U_t = U_t^0$, $t_0 < t < T$ are of the same type, more precisely, are isometric.

If we introduce an operator A satisfying the condition

$$A^*A = B \quad (2.2)$$

(for example, $A = B^{1/2}$ —the positive square root of the positive operator B can be taken), then we shall have

$$E(\xi, u)\overline{(\xi, v)} = (Bu, v) = (Au, Av), \quad u, v \in U^0,$$

from which it is seen that the families $H_t(\xi)$ and $\overline{AU_t^0}$, $t_0 < t < T$, are isometric, and the question posed above can be reformulated as the following: under which conditions imposed on the operator A will the families

$$H_t = \overline{AU_t^0} \quad \text{and} \quad U_t = \overline{U_t^0}, \quad t_0 < t < T \quad (2.3)$$

be isometric [compare (1.4) and *et seq.*]?

Generally, considering the question about a type of the family of subspaces $H_t(\xi)$, $t_0 < t < T$, appearing in “observing” an ordinary random process $\xi(t)$, $t_0 < t < T$, or, comparing $H_t(\xi)$, $t_0 < t < T$, with some “standard” family U_t , $t_0 < t < T$, in any Hilbert space U , it is, as a rule, possible to go over to a “generalized process” (ξ, u) , $u \in U^0$, described above, in the Hilbert space U with a given family of subspaces U_t^0 , $t_0 < t < T$, such that $H_t(\xi)$ coincides with the closure of the subspace of all values (ξ, u) , $u \in U_t^0$, and $U_t = \overline{U_t^0}$, $t_0 < t < T$.

We shall carry out this simple reduction for an infinite-dimensional process with components denoted by $\{\xi(t), x\}$, $t_0 < t < T$, indexed by $x \in R$, where R is a separable Hilbert space with the scalar product $\{x, y\}$, $x, y \in R$. We shall assume here that the correlation function $B(t, s)$, $t_0 < t, s < T$, of such

a process, defined from the relation

$$E \{ \xi(t), x \} \overline{\{ \xi(s), y \}} = \{ B(t, s) x, y \}, \quad x, y \in R, \quad (2.4)$$

is a weakly continuous operator function in R .

Introduce the linear space U^0 of all linear combinations in R of the form

$$u(t) = \sum_k x_k c_k(t), \quad t_0 < t < T,$$

where the coefficients x_k are elements in the Hilbert space R , and the numerical functions $c_k(t)$ all vanish outside a finite interval and

satisfy the condition $\int_{t_0}^T |c_k(t)|^2 dt < \infty$. Denote by U_t^0 the subspace

spanned by the functions $u = u(s)$, $t_0 < s < T$, of the type indicated, satisfying the additional condition $u(s) = 0$, with $s > t$. Introduce also the Hilbert space $U = L^2(R)$ of all measurable functions $u = u(t)$, $t_0 < t < T$, with values in R , for which

$\int_{t_0}^T \|u(t)\|^2 dt < \infty$, with scalar product of the elements $u, v \in U$

defined by

$$(u, v) = \int_{t_0}^T \{u(t), v(t)\} dt. \quad (2.5)$$

Obviously, the closure $\overline{U}_t^0$ in U is the subspace U_t of all the functions $u = u(s)$, $t_0 < t < T$, from $U = \overline{U}^0$, satisfying the condition

$$u(s) = 0 \quad \text{with } s > t, \quad (2.6)$$

and the structure of the family of the subspaces U_t , $t_0 < t < T$, can be taken as given standard.

Define a generalized random process $(\xi, u), u \in U$, setting

$$(\xi, u) = \int_{t_0}^T \{\xi(t), u(t)\} dt = \sum_k \int_{t_0}^T \{\xi(t), x_k\} \overline{c_k(t)} dt \quad (2.7)$$

with $u(t) = \sum_k x_k c_k(t) \in U^0$.

Obviously, the closed linear hull $H_t(\xi)$ of all the values $\{\xi(s), x\}$, $x \in R$; $t_0 < s \leq t$, coincides with the closure of the subspace $H_t^0(\xi)$ of all the values (ξ, u) , $u \in U_t^0$ (recall that we consider the correlation function $B(s, t)$ to be weakly continuous, and this is equivalent to continuity in mean square of all the components $\{\xi(t), x\}$, $t_0 < t < T$, of the primary process).

Next, from the formulas (2.4), (2.7) it can be easily deduced that

$$\begin{aligned} \mathbf{E}(\xi, u) \overline{(\xi, v)} &= \\ &= \int_{t_0}^T \int_{t_0}^T \{B(s, t) u(s), v(t)\} ds dt, \quad u, v \in U^0, \end{aligned} \quad (2.8)$$

and if the condition

$$\int_{t_0}^T \int_{t_0}^T \|B(s, t)\|^2 ds dt < \infty \quad (2.9)$$

is assumed satisfied, then the generalized process (ξ, u) , $u \in U^0$, can be extended continuously to the whole Hilbert space $U = L^2(R)$ and the bounded positive operator B with the kernel $B(s, t)$:

$$Bu(t) = \int_{t_0}^T B(t, s) u(s) ds, \quad u \in U, \quad (2.10)$$

will be a correlation operator for $\xi(u)$, $u \in U$.

As was noted above [see (2.3)], the question whether the family $H_t(\xi)$, $t_0 < t < T$, would be isometric to the standard family U_t , $t_0 < t < T$, is reduced to an analogous question for the family $H_t = AU_t$, $t_0 < t < T$, where $A = B^{1/2}$ is the square root of the positive operator B .

Let us now turn to another major case—the infinite-dimensional stationary process $\xi(t)$, $-\infty < t < \infty$, with the components $\{\xi(t), x\}$, $x \in R$, being stationarily related (usual-sense) processes. The stationarity condition implies that the correlation function $B(t, s)$ —see (2.4)—depends only on the difference $t - s$.

Suppose that there is a spectral density f_λ , $-\infty < \lambda < \infty$, a measurable operator function in the Hilbert space R , positive-definite for each λ , such that

$$\begin{aligned} E \{ \xi(t), x \} \overline{ \{ \xi(s), y \} } &= \{ B(t-s)x, y \} = \\ &= \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \{ f_\lambda x, y \} d\lambda, \quad x, y \in R. \end{aligned} \quad (2.11)$$

Suppose for simplicity that the function f_λ is bounded ($\|f_\lambda\| < C$). Taking the corresponding generalized process (ξ, u) , $u \in U^0$ [setting $t_0 = -\infty$ and $T = \infty$ in (2.7)], we shall have

$$E(\xi, u) \overline{(\xi, v)} = \int_{-\infty}^{\infty} \{ f_\lambda \tilde{u}(\lambda), \tilde{v}(\lambda) \} d\lambda, \quad (2.12)$$

where $\tilde{u}(\lambda)$ designates the Fourier transform of the corresponding function $u \in U^0$, namely,

$$\begin{aligned} \tilde{u}(\lambda) &= \int_{-\infty}^{\infty} e^{i\lambda t} u(t) dt = \\ &= \sum_k x_k \int_{-\infty}^{\infty} e^{i\lambda t} c_k(t) dt, \quad -\infty < \lambda < \infty, \end{aligned} \quad (2.13)$$

with $u(t) = \sum_k x_k c_k(t)$, $-\infty < t < \infty$.

It will be more convenient to go over to a new parametric space if we take instead of U^0 the space $\tilde{U}^0$ of all functions $\tilde{u}(\lambda)$ which are Fourier transforms of $u(t) \in U^0$ and set

$$(\xi, \tilde{u}) = (\xi, u), \quad \tilde{u} \in \tilde{U}^0. \quad (2.14)$$

From the formula (2.12) it is seen that the generalized process $(\xi, \tilde{u})$, $\tilde{u} \in \tilde{U}^0$, has for its correlation operator a *multiplier* operator of multiplication by the spectral density f_λ :

$$[B\tilde{u}](\lambda) = f_\lambda \tilde{u}(\lambda), \quad -\infty < \lambda < \infty. \quad (2.15)$$

Denote by $\tilde{U}_t^0$ the subspace in the space $\tilde{U}^0$ formed by all elements $\tilde{u} \in \tilde{U}^0$, obtained by means of the Fourier transform [see (2.13)] of the functions $u = u(t)$ from the corresponding subspace U_t^0 , $-\infty < t < \infty$.

As is well-known, the Fourier transform is a unitary operator in $L^2(R)$, and, consequently, the families U_t^0 and $\tilde{U}_t^0$, $-\infty < t < \infty$, are unitarily isomorphic. As the formula (2.13) indicates, the families of the subspaces $H_t^0(\xi)$ and $A\tilde{U}_t^0$, $-\infty < t < \infty$, will be also unitarily isomorphic, where $A = B^{1/2}$ is the *multiplier* operator (by $f_\lambda^{1/2}$):

$$[A\tilde{u}](\lambda) = f_\lambda^{1/2} \tilde{u}(\lambda), \quad -\infty < \lambda < \infty. \quad (2.16)$$

Therefore, the question of conditions to be imposed on the spectral density f_λ under which the family $H_t(\xi)$ will be of the same type as that of the standard family U_t , $-\infty < t < \infty$, described above [see (2.6)], is a particular case of the general question of isometry of families of the form $\overline{AU}_t^0$, $t_0 < t < T$, and $U_t = \overline{U}_t^0$, $t_0 < t < T$, for the generalized process (ξ, u) , $u \in U^0$, on the Hilbert space U^0 with the correlation operator $B = A^*A$.

Let us consider now an arbitrary generalized process (ξ, u) , $u \in U^0$, with the correlation operator B in a Hilbert space $U = \overline{U}^0$ with a prescribed family of subspaces U_t^0 , $t_0 < t < T$, and consider the corresponding family $H_t(\xi)$, $t_0 < t < T$ ($H_t(\xi)$ denotes the closed linear hull of all values $\xi(u)$, $u \in U_t^0$). Let $U_t = \overline{U}_t^0$ and

assume that the space U coincides with the closure of the union of all the subspaces U_t , $t_0 < t < T$.

We shall call the process (ξ, u) , $u \in U^0$, a regular process if the families $H_t(\xi)$ and U_t , $t_0 < t < T$, are isometric:

$$H_t(\xi) = XU_t, \quad t_0 < t < T, \quad (2.17)$$

where X is a unitary operator on U into the space $H(\xi)$ —the closed linear hull of all the values $\xi(u)$, $u \in U^0$.

We shall say that the operator B is factorizable if it is representable as

$$B = C^*C, \quad (2.18)$$

where C is a linear operator on U^0 into U , such that

$$\overline{CU_t^0} = U_t, \quad t_0 < t < T. \quad (2.19)$$

Theorem. *The process (ξ, u) , $u \in U^0$, is regular if and only if its correlation operator B can be factorized.*

PROOF. As was noted above, the family $H_t(\xi)$, $t_0 < t < T$, is isometric to any family of subspaces of the form $\overline{AU_t^0}$, $t_0 < t < T$, where A is the linear operator defined on U^0 and satisfying the condition $A^*A = B$. Hence, if the process (ξ, u) , $u \in U^0$, is regular, the family of the subspaces $\overline{AU_t^0}$, $t_0 < t < T$, where $A = B^{1/2}$ is the positive square root of the correlation operator B , is isometric to the family $U_t = \overline{U_t^0}$, $t_0 < t < T$, i.e., there exists a unitary operator X on $U = \overline{U^0}$ into $\overline{AU^0}$, such that $\overline{AU_t^0} = XU_t$, $t_0 < t < T$. Obviously, the operator $C = X^{-1}A$ satisfies the conditions (2.18), (2.19).

On the other hand, if the correlation operator B can be factorized [i.e., there exists an operator C satisfying the conditions (2.18), (2.19)], then the subspaces $\overline{AU_t^0}$ with $A = C$ coincide with U_t , and, as shown above, regularity is equivalent to the isometry of the families $\overline{AU_t^0}$ and U_t , $t_0 < t < T$ (for any A , $A^*A = B$). Consequently, if B is factorizable, then the process (ξ, u) , $u \in U^0$, is regular.

As the examples above demonstrate, not every operator B can be factorized.

We wish to note the following about factorization.

Suppose that a positive operator B on a Hilbert space U is representable as $B = C^*C$. Such a representation is certainly not unique; for instance, we have as well $B = C_1^* \cdot C_1$ for $C_1 = XC$, where X is an arbitrary unitary operator on U . On the other hand, if for any C and C_1 the condition $B = C^*C = C_1^*C_1$ is satisfied, then the operator X defined on the subspace CU^0 by means of the equality $XCu = C_1u$, $u \in U^0$, will be isometric:

$$\begin{aligned} (XCu, X Cv) &= (C_1u, C_1v) = \\ &= (C_1^*C_1u, v) = (C^*Cu, v) = (Cu, Cv), \quad u, v \in U. \end{aligned}$$

Under the condition $\overline{CU_t^0} = \overline{C_1U_t^0} = U_t$ for all t [see (2.19)] we have $\overline{CU^0} = \overline{C_1U^0} = U$, and, consequently, the operator X can be extended uniquely to a unitary operator on the whole space U . Furthermore, since for any $u \in U_t$ there would be a sequence $u_n \in U_t^0$, $n = 1, 2, \dots$, such that $u = \lim Cu_n$, we have:

$$Xu = \lim XCu_n = \lim C_1u_n \in U_t.$$

Therefore, under the condition (2.19) the factorizing operator C in the relation (2.18) is defined uniquely up to the unitary multiplier X satisfying the condition

$$XU_t = U_t, \quad t_0 < t < T. \quad (2.20)$$

For the family $U_0, t_0 < t < T$, prescribed, with the structural types $dF_1(t) \succ \dots \succ dF_N(t)$ the following standard model can be exploited: U is the space of the vector functions $u(t) = \{u_k(t)\}_1^N$, $t_0 < t < T$, with the components $u_k(t)$,

$$u_k(t) = 0 \quad \text{for} \quad dF_k(t)/dF_1(t) = 0, \quad k = 1, \dots, N,$$

in which the scalar product of the elements $u(t) = \{u_k(t)\}_1^N$ and

$v(t) = \{v_k(t)\}_1^N$ is defined by the formula

$$(u, v) = \int_{t_0}^T \left[\sum_{k=1}^N u_k(t) \overline{v_k(t)} \right] dF_1(t);$$

the family U_t , $t_0 < t < T$, in the function space U is realized as a family of subspaces

$$U_t = \{u(s), t_0 < s < T: u(s) = 0 \text{ for } s > t\}.$$

(For this, see Sec. 2, Chap. I.) Denote by E the N -dimensional vector space with the usual scalar product $\sum_{k=1}^N x_k \bar{y}_k$ of the vectors $x = \{x_k\}_1^N$, $y = \{y_k\}_1^N \in E$ and by E_t the subspace spanned by the vectors $x = \{x_k\}_1^N$ with the components

$$x_k = 0 \quad \text{with} \quad dF_k(t)/dF_1(t) = 0.$$

Let X be a unitary operator on U satisfying the condition (2.20). Then for all t

$$\begin{aligned} \int_{t_0}^t \sum_{k=1}^N (Xu)_k(s) \cdot \overline{(Xv)_k(s)} dF_1(s) = \\ = \int_{t_0}^t u_k(s) \cdot \overline{v_k(s)} dF_1(s), \quad u, v \in U. \end{aligned}$$

It is seen that for fixed s the mapping

$$\{u_k(s)\}_1^N \rightarrow \{(Xu)_k(s)\}_1^N,$$

defined for almost all s , is unitary in the subspace $E_s \subseteq E$. From this we can readily deduce the general form of the unitary operator X satisfying the condition (2.20) in the function space U :

$$(Xu)(t) = X_t u(t), \quad t_0 < t < T, \quad (2.21)$$

where X_t , for fixed t , is the arbitrary unitary operator in the subspace E_t of the vector space E (it is only necessary that the operator function X_t , $t_0 < t < T$, be measurable).

§3. A Theorem on Factorization

In this section our objective is to prove the following:*

Theorem. *The positive (invertible) operator B on the Hilbert space U of the form $B = I - G$, where G —a Hilbert-Schmidt operator—can be factorized as*

$$B = C^* C \quad (\overline{CU_t} = U_t, \quad t_0 < t < T) \quad (3.1)$$

with respect to any family of the subspaces U_t , $t_0 < t < T$.

As a preliminary, we shall note that the operator B permits the factorization (3.1), if it is representable as

$$B = C_- \cdot C_+, \quad (3.2)$$

where the operators C_- and C_+ are invertible and satisfy the conditions

$$C_+ U_t = U_t, \quad C_- U_t^\perp \subseteq U_t^\perp; \quad t_0 < t < T;$$

here the “adjoint” family of the subspaces

$$U_t^\perp = U \ominus U_t, \quad t_0 < t < T,$$

*Compare with the discussion presented in I. C. Gohberg and M. G. Krein, “Theory and Application of Volterra Operators in Hilbert Space,” Am. Math. Society Translation, 1970 (Theorems 6.2, Chap. IV and 10.1, Chap. I).

is invariant with respect to the operator C_- if and only if the family U_t , $t_0 < t < T$, itself is invariant with respect to the adjoint operator C_-^* : $C_-^* U_t \subseteq U_t$.

Indeed, in the case of the positive operator B the operator

$$\begin{aligned} D &= (C_+^{-1})^* B C_+^{-1} = (C_+^*)^{-1} B C_+^{-1} = \\ &= (C_+^*)^{-1} C_+^* C_- C_+^{-1} = C_- C_+^{-1}, \end{aligned}$$

as well, will be positive, for which, obviously, $DU_t \subseteq U_t$. Since the operator D is invertible, we actually have $DU_t = U_t$.

If we introduce an operator $D^{1/2}$ —the positive root of D —then we shall obtain that

$$B = C_+^* D C_+ = C^* C,$$

where the operator $C = D^{1/2} C_+$ satisfies the condition

$$C U_t = U_t, \quad t_0 < t < T.$$

We shall show next that the invertible operator $B = I - G$ can be factorized with the aid of the operators C_+ and $C_- = B C_+^{-1}$ [see (3.2)], setting $C_+ = (I + G_+)^{-1}$, where the operator G_+ is defined as some integral

$$G_+ = \int \int_{s < t} (I - Q_s G Q_t)^{-1} dQ_s G dQ_t \quad (3.3)$$

over “the operator measure” $dQ_s G dQ_t$ (here Q_t are projectors on the subspaces U_t , $t_0 < t < T$).

Note immediately that since B is a positive invertible operator,

$$\sup_{\|u\|=1} (Gu, u) = 1 - \inf_{\|u\|=1} (Bu, u) = r < 1$$

and for any projector Q

$$\inf_{\|u\|=1} ([I - QGQ]u, u) = 1 - \sup_{\|u\|=1} (QGQu, u) \geq 1 - r > 0,$$

and, consequently, $I - QGQ$ is a positive invertible operator, with

$$\|I - QGQ\|^{-1} \leq \frac{1}{1 - r}. \quad (3.4)$$

Choose the finite decomposition $\gamma = \{t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$. Setting $\Delta Q_{t_k} = Q_{t_{k+1}} - Q_{t_k}$, any operator A can be represented as

$$\begin{aligned} A &= \left(\sum_{k=0}^n \Delta Q_{t_k} \right) A \left(\sum_{j=0}^n \Delta Q_{t_j} \right) = \\ &= \sum_{k < j} \Delta Q_{t_k} A \Delta Q_{t_j} + \sum_{k \geq j} \Delta Q_{t_k} A \Delta Q_{t_j}, \end{aligned}$$

where

$$S(A) = \sum_{k < j} \Delta Q_{t_k} A \Delta Q_{t_j}$$

possesses the property that

$$\begin{aligned} S(A) U_{t_k} &= \sum_{j=1}^n (Q_{t_j} A \Delta Q_{t_j}) U_{t_k} = \\ &= \sum_{j=1}^{k-1} (Q_{t_j} A \Delta Q_{t_j}) U_{t_k} \subseteq U_{t_{k-1}}, \quad k = 1, \dots, n+1, \end{aligned} \quad (3.5)$$

and

$$S(A) U_{t_0} = 0,$$

and the supplementary part $S'(A) = A - S(A)$,

$$S'(A) = \sum_{k \geq j} \Delta Q_{t_k} A \Delta Q_{t_j},$$

is such that

$$\begin{aligned} S'(A) U_{t_j}^\perp &= \sum_{k=0}^n (\Delta Q_{t_k} A Q_{t_{k+1}}) U_{t_j}^\perp = \\ &= \sum_{k=j}^n (\Delta Q_{t_k} A Q_{t_{k+1}}) U_{t_j}^\perp \subseteq U_{t_j}^\perp, \quad j = 0, \dots, n. \end{aligned} \quad (3.6)$$

Let us see what the "operation of truncation" $S(A)$ yields for the operator

$$A = G(I + G_+),$$

where $G = I - B$ and

$$G_+ = \sum_{k < j} (I - Q_{t_j} G Q_{t_j})^{-1} \Delta Q_{t_k} G \Delta Q_{t_j}. \quad (3.7)$$

We have

$$\begin{aligned} S(G \cdot G_+) &= \sum_{j=1}^n Q_{t_j} (G \cdot G_+) \Delta Q_{t_j} = \\ &= \sum_{j=1}^n Q_{t_j} G \sum_{k=1}^n (I - Q_{t_k} G Q_{t_k})^{-1} Q_{t_k} G \Delta Q_{t_k} \Delta Q_{t_j} = \\ &= \sum_{k=1}^n Q_{t_k} G (I - Q_{t_k} G Q_{t_k})^{-1} Q_{t_k} G \Delta Q_{t_k}, \end{aligned}$$

and since each of the subspaces U_{t_k} is invariant with respect to the operators $I - Q_{t_k} G Q_{t_k}$, $(I - Q_{t_k} G Q_{t_k})^{-1}$ and the projector Q_{t_k} commutes with them, we have

$$S(G \cdot G_+) = \sum_{k=1}^n Q_{t_k} G Q_{t_k} (I - Q_{t_k} G Q_{t_k})^{-1} Q_{t_k} G \Delta Q_{t_k}.$$

As a result we obtain

$$\begin{aligned}
S[G(I + G_+)] &= S(G) + S(G \cdot G_+) = \\
&= \sum_{k=1}^n [I + Q_{t_k} G Q_{t_k} (I - Q_{t_k} G Q_{t_k})^{-1}] Q_{t_k} G \Delta Q_{t_k} = \\
&= \sum_{k=1}^n (I - Q_{t_k} G Q_{t_k})^{-1} Q_{t_k} G \Delta Q_{t_k} = G_+.
\end{aligned}$$

Thus, the operator G_+ is a "truncation" of the operator $A = G(I + G_+)$:

$$S[G(I + G_+)] = G_+.$$

It can be easily verified that

$$S(A)^n = 0 \quad (3.8)$$

[compare with (3.5)], and, consequently, the spectrum of the truncated operator $G_+ = S(A)$ consists of only the point $\lambda = 0$, so that the operator $(I + G_+)$ is invertible. If we take $G_- = S'(A)$ for $A = G(I + G_+)$, then we shall have

$$G_+ + G_- = G(I + G_+)$$

and

$$\begin{aligned}
(I - G)(I + G_+) &= I + G_+ - G(I + G_+) = \\
&= I + G_+ - G_+ - G_- = I - G_-,
\end{aligned}$$

from which we find

$$(I - G) = (I - G_-)(I + G_+)^{-1}. \quad (3.9)$$

The subspaces U_{t_k} , $k = 0, 1, \dots, n+1$, are invariant with respect to the truncated operator $G_+ = S[G(I + G_+)]$ [see (3.5)] and, consequently, with respect to the operators $I + G_+$, $(I + G_+)^{-1}$. In turn, the subspaces $U_{t_k}^\perp$, $k = 0, 1, \dots, n+1$, are invariant with respect to the operators $G_- = S'[G(I + G_+)]$

and $I - G_-$ [see (3.6)]. Therefore, the relation (3.9) provides the factorization of the operator $B = I - G$ with the aid of $C_+ = (I + G_+)^{-1}$ and $C_- = I - G_-$ with respect to the discrete chain $U_{t_k}, t_0 < t_1 < \dots < t_{k+1} = T$ [see (3.2)].

The expression (3.7) for the operator $G_+ = G_+^{(n)}$ resembles an integral sum, and if upon further decomposition $\gamma = \{t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$ there exists a limit operator $G_+ = \lim_{n \rightarrow \infty} G_+^{(n)}$, then the latter should be naturally denoted as in the

formula (3.3). Obviously, if there exists in the limit (in the sense of strong convergence) a bounded operator G_+ such that $I + G_+$ has the inverse $C_+ = (I + G_+)^{-1}$, then there also exists a limit operator

$$C_- = \lim_{n \rightarrow \infty} B(I + G_+^{(n)}) = B(I + G_+)$$

and the equality $B = C_- \cdot C_+$ yields the factorization with respect to the complete family $U_t, t_0 < t < T$.

Indeed, any element $u \in U_t$ is the limit of the elements $u_k \in U_{t_k}$ with $t_k \rightarrow t - 0$, and, hence

$$G_+ u_k = \lim_{n \rightarrow \infty} G_+^{(n)} u_k \in U_{t_k} \subseteq U_t, \quad G_+ u = \lim_{t_k \rightarrow t-0} G_+ u_k \in U_t,$$

so that each of the subspaces U_t is invariant with respect to the operators $I + G_+$, $(I + G_+)^{-1}$. Analogous considerations can be applied to the operator $C_- = \lim_{n \rightarrow \infty} C_-^{(n)}$, where $C_-^{(n)} = B(I + G_+^{(n)})$

satisfies the condition $C_-^{(n)} U_{t_k}^\perp \subseteq U_{t_k}^\perp$ for any t_k from the corresponding decomposition $\gamma^{(n)} = \{t_0 < t_1 < \dots < t_n < T\}$.

Let us show now that there exists a uniform limit $\lim_{n \rightarrow \infty} G_+^{(n)} = G_+$ in the case where

$$G = I - B$$

is a Hilbert-Schmidt operator with a finite "trace norm":

$$|G|^2 = \text{Sp } G^* G = \sum_p \|G u_p\|^2 = \sum_{p,q} |(G u_p, u_q)|^2 < \infty \quad (3.10)$$

(where $u_1, u_2, \dots$ denote some orthonormalized basis in a Hilbert space U), and, the limit operator G_+ , as well as the operators $G_+^{(n)}$, has the only spectrum point $\lambda = 0$, so that the operator $(I + G_+)$ will be invertible.

For any decompositions $\gamma^{(m)} = \{t_0 < s_1 < \dots < s_m < s_{m+1} < T\}$ and $\gamma^{(n)} = \{t_0 < t_1 < \dots < t_n < t_{n+1} = T\}$ such that $\gamma^{(m)} \subseteq \gamma^{(n)}$ (each decomposition interval $[s_i, s_{i+1})$ coincides with the join of some intervals $[t_j, t_{j+1})$), the difference $G_+^{(n)} - G_+^{(m)}$ can be represented as

$$G_+^{(n)} - G_+^{(m)} = \sum_{j=1}^n (F_{t_j} Q_{t_j} - F_{s(j)} Q_{s(j)}) G \Delta Q_{t_j},$$

where $F_t = (I - Q_t G Q_t)^{-1}$, and $s(j)$ designates the left end of the interval from the decomposition $\gamma^{(m)}$ containing the point $t_j \in \gamma^{(n)}$ ($s_i \leq t_j < s_{i+1}$ with $s(j) = s_i$)—see formula (3.7).

Note that for any operators A_j

$$\left| \sum_j A_j G \Delta Q_{t_j} \right|^2 = \sum_j |A_j G \Delta Q_{t_j}|^2. \quad (3.11)$$

This equality can be readily obtained if we choose complete orthonormal systems in orthogonal subspaces $\Delta Q_{t_j} U = U_{t_{j+1}} \ominus U_{t_j}$, $j = 0, 1, \dots, n$, and take their union $u_1, u_2, \dots$ as the basis for the whole space U .

Indeed, $A_j G \Delta Q_{t_j} u_p = 0$ with $j \neq j(p)$, if $u_p \in \Delta Q_{t_{j(p)}}$, from which it follows that

$$\left\| \sum_j A_j G \Delta Q_{t_j} u_p \right\|^2 = \sum_j \|A_j G \Delta Q_{t_j} u_p\|^2$$

and

$$\begin{aligned} \left| \sum_j A_j G \Delta Q_{t_j} \right|^2 &= \sum_p \left\| \sum_j A_j G \Delta Q_{t_j} u_p \right\|^2 = \\ &= \sum_p \sum_j \|A_j G \Delta Q_{t_j} u_p\|^2 = \sum_j \left[\sum_p \|A_j G \Delta Q_{t_j} u_p\|^2 \right] = \\ &= \sum_j |A_j G \Delta Q_{t_j}|^2. \end{aligned}$$

Furthermore, for any bounded A_1, A_2 the following relations hold true:

$$\begin{aligned} |A_1 G A_2|^2 &= \sum_{p, q} |(A_1 G A_2 u_p, u_q)|^2 = \\ &= \sum_{p, q} |(A_2^* G A_1^* u_q, u_p)|^2 = |A_2^* G A_1^*|^2 \end{aligned}$$

and

$$\begin{aligned} |A_1 G A_2|^2 &= \sum_p \|A_1 G A_2 u_p\|^2 \leq \\ &\leq \|A_1\|^2 \cdot \sum_p \|G A_2 u_p\|^2 = \|A_1\|^2 \cdot |G A_2|^2 = \\ &= \|A_1\|^2 \cdot |A_2^* G|^2 \leq \|A_1\|^2 \cdot \|A_2^*\|^2 \cdot |G|^2 = \\ &= \|A_1\|^2 \cdot \|A_2\|^2 \cdot |G|^2. \quad (3.12) \end{aligned}$$

By using the relations (3.11), (3.12), we obtain readily that

$$\begin{aligned} |G_+^{(n)} - G_+^{(m)}|^2 &= \sum_{j=1}^n |(F_{t_j} Q_{t_j} - F_{s(j)} Q_{s(j)}) G \Delta Q_{t_j}|^2 = \\ &= \sum_{j=1}^n |(F_{t_j} - F_{s(j)}) Q_{s(j)} G \Delta Q_{t_j} + F_{t_j} (Q_{t_j} - Q_{s(j)}) G \Delta Q_{t_j}|^2 \leq \\ &\leq 2 \sum_{j=1}^n (\|F_{t_j} - F_{s(j)}\|^2 \|Q_{s(j)}\|^2 |G \Delta Q_{t_j}|^2 + \\ &\quad + \|F_{t_j}\|^2 |\Delta Q_{s(j)} G \Delta Q_{t_j}|^2, \end{aligned}$$

where $\Delta Q_{s(j)} = Q_{t_j} - Q_{s(j)}$ ($s(j) \leq t_j$). Since $\|Q_{s(j)}\| = 1$ and $\|F_{t_j}\| \leq \frac{1}{1-r}$ [see the inequality (3.4)], we have

$$\begin{aligned} |G_+^{(n)} - G_+^{(m)}|^2 &\leq 2 \sum_{j=1}^n \|F_{t_j} - F_{s(j)}\|^2 |G \Delta Q_{t_j}|^2 + \\ &\quad + \frac{2}{(1-r)^2} \sum_{j=1}^n |\Delta Q_{s(j)} G \Delta Q_{t_j}|^2. \quad (3.13) \end{aligned}$$

According to the general formula (3.11) the function $\lambda(\Delta t) = |G\Delta Q_t|^2$ on the semi-intervals $\Delta t = [t, s)$ is additive. Having chosen the orthonormalized eigenfunction basis: $u_1, u_2, \dots$ ($Gu_p = \lambda_p u_p$, $\sum_p \lambda_p^2 < \infty$) corresponding to the operator G , we have

$$\begin{aligned} |G\Delta Q_t|^2 &= |\Delta Q_t G|^2 = \sum_p \|\Delta Q_t G u_p\|^2 = \\ &= \sum_p \lambda_p^2 \|\Delta Q_t u_p\|^2 \rightarrow 0 \quad \text{as } t \rightarrow s, \end{aligned} \quad (3.14)$$

since the projectors Q_s are left-continuous: $\Delta Q_t = Q_s - Q_t \rightarrow 0$ with $t \rightarrow s$, and for any fixed element $u \in U$ its projection $\Delta Q_t u$ onto the subspace $\Delta U_t = U_s \ominus U_t$ is such that $\|\Delta Q_s u\| \rightarrow 0$.

Therefore,

$$\lambda(\Delta t) = |G\Delta Q_t|^2$$

is not only an additive but also a continuous function on the sets $\Delta t = [t, s)$, and, consequently, it can be extended to a bounded Borel measure on the interval $[t_0, T)$.

Obviously, the same properties hold for the set function

$$\mu(\Delta s \times \Delta t) = |\Delta Q_s G \Delta Q_t|^2,$$

defined on the rectangles $\Delta s \times \Delta t = [s_1, s_2) \times [t_1, t_2)$; namely, from the general function (3.11) it follows that $\mu(\Delta s \times \Delta t)$ is additive, and from the estimate (3.14) it follows that $\mu(\Delta s \times \Delta t)$ is continuous, and, consequently, can be extended to a bounded Borel measure on the square $[t_0, T) \times [t_0, T)$.

Let $f(t) = \mu\{[t_0, t) \times [t_0, t)\}$, $t_0 < t < T$. For the operators $F_t = (I - Q_t G Q_t)^{-1}$ we have

$$\begin{aligned} F_s^{-1} - F_t^{-1} &= Q_s G Q_s - Q_t G Q_t, \\ |F_s^{-1} - F_t^{-1}|^2 &= f(t) - f(s), \quad s \leq t, \end{aligned}$$

and

$$\begin{aligned} |F_t - F_s|^2 &= |F_t (F_s^{-1} - F_t^{-1}) F_s|^2 \leq \\ &\leq \frac{1}{(1-r)^2} |F_s^{-1} - F_t^{-1}|^2 = \frac{1}{(1-r)^2} [f(t) - f(s)]. \end{aligned}$$

It is seen that the first member on the right-hand side of the inequality (3.13) satisfies the condition

$$\begin{aligned} \sum_{j=1}^n \|F_{t_j} - F_{s(j)}\|^2 |G \Delta Q_{t_j}|^2 &\leq \frac{1}{(1-r)^2} \sum_{j=1}^n [f(t_j) - f(s(j))] = \\ &= \frac{1}{(1-r)^2} (I_n - I_m), \end{aligned}$$

where $I_m = \sum_{i=0}^m f(s(i)) \Delta$, $I_n = \sum_{j=0}^n f(t_j) \lambda(\Delta t_j)$ are "left-integral sums" for the nondecreasing function $f(t)$, $t_0 < t < T$. The sequence of integral sums I_n , $n = 1, 2, \dots$ is bounded and non-decreasing (with finer decompositions), so that there exists $\lim_{n \rightarrow \infty} I_n$, and, consequently, with $m, n \rightarrow \infty$

$$\sum_{j=1}^n \|F_{t_j} - F_{s(j)}\|^2 |G \Delta Q_{t_j}|^2 \leq \frac{1}{(1-r)^2} (I_n - I_m) \rightarrow 0.$$

Next, introduce the set

$$\Delta_h = \{s, t: s < t < s + h\}.$$

The union $\bigcup_{j=1}^n [s(j), t_j) \times [t_j, t_{j+1})$ is contained in Δ_h with $h = \max_i |s_{t+1} - s_i|$, when $t_{j+1} - s(j) \leq h$. Consequently, for the second sum in the right-hand side of the inequality (3.13) we have:

$$\sum_{j=1}^n |\Delta Q_{s(j)} G \Delta Q_{t_j}|^2 \leq \mu(\Delta_h) \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

From the estimates obtained it follows that for the sequence of decompositions $\gamma^{(n)}$, $n = 1, 2, \dots$, imbedded in each other, the corresponding operators $G_+^{(n)}$ are such that $\|G_+^{(n)} - G_+^{(m)}\|^2 \rightarrow 0$ as $n, m \rightarrow \infty$, and, consequently, there exists a limit operator G_+ ;

$$\|G_+^{(n)} - G_+\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

It remains to show that the limit operator G_+ , as well as the operators $G_+^{(n)} = S(G(I + G_+^{(n)}))$ [see (3.8)], have the only spectrum point $\lambda = 0$.

Being a completely-continuous operator (and even a Hilbert-Schmidt operator), G_+ can have no more than a countable number of points in the spectrum, with limit point $\lambda = 0$, only. Therefore, any point $\lambda \neq 0$ has a neighborhood entirely belonging to the resolvent set of the operator G_+ with the exception of λ , and on any closed contour Γ of this neighborhood the resolvent $R_z = (G_+ - zI)^{-1}$ is defined which is continuous and bounded on Γ . Under the condition of uniform convergence $G_+^{(n)} \rightarrow G_+$ for all sufficiently large n

$$\begin{aligned} R_z^{(n)} &= (G_+^{(n)} - zI)^{-1} = R_z[I - R_z(G_+ - G_+^{(n)})]^{-1} = \\ &= R_z \sum_{k=0}^{\infty} R_z^k (G_+ - G_+^{(n)})^k \end{aligned}$$

and $R_z^{(n)} \rightarrow R_z$ uniformly over $z \in \Gamma$ as $n \rightarrow \infty$, so that

$$\int_{\Gamma} z^k R_z dz = \lim_{n \rightarrow \infty} \int_{\Gamma} z^k R_z^{(n)} dz = 0 \quad \text{for all } k \geq 0,$$

since $z = 0$ is the only singular point of the analytic functions $R_z^{(n)}$ and $\int_{\Gamma} z^k R_z^{(n)} dz = 0$. Therefore, $\int_{\Gamma} z^k R_z dz = 0$, which implies analyticity of R_z at the point λ .

Consequently, $\lambda = 0$ is the only point in the spectrum of the operator G_+ .

To conclude, we have established that the positive invertible operator $B = I - G$, where G is a Hilbert-Schmidt operator [see the condition (3.10)], permits a representation $B = C_- \cdot G_+$ of the type (3.2); as C_+ the operator $C_+ = (I + G_+)^{-1}$ can be taken where $G_+ = \lim G_+^{(n)}$ is the limit in the "trace norm" of the Hilbert-Schmidt operators $G_+^{(n)}$ defined by the formula (3.7). From this, as noted before, the possibility of factorization with respect to any family of subspaces U_t , $t_0 < t < T$, follows.

Chapter III

REGULAR STATIONARY PROCESSES

§ 1. A Structural Type of a Regular Stationary Process

Let us discuss now the model described above of the stationary process $\xi(t)$, $-\infty < t < \infty$, with the stationarily related components $\{\xi(t), x\}$, $x \in R$, each of which corresponds to some $x \in R$, where R is a separable Hilbert space with scalar product $\{x, y\}$, $x, y \in R$. We shall assume that there exists a spectral density f_λ , $-\infty < \lambda < \infty$, a measurable operator function on the Hilbert space R (for fixed λ this function being a positive operator on R), such that

$$\begin{aligned} E \{ \xi(t), x \} \cdot \overline{\{ \xi(s), y \}} &= \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \{ f_\lambda x, y \} d\lambda = \\ &= \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \{ f_\lambda^{1/2} x, f_\lambda^{1/2} y \} d\lambda, \quad x, y \in R, \end{aligned} \quad (1.1)$$

where $f_\lambda^{1/2}$ designates the (positive) "square root" of f_λ , more precisely, $f_\lambda^{1/2}$, $-\infty < \lambda < \infty$, is the operator function satisfying the condition $(f_\lambda^{1/2})^2 = f_\lambda$.

Introduce the Hilbert space $L^2(R)$ of square-integrable functions $x(\lambda)$, $-\infty < \lambda < \infty$, with values in R , and scalar product in $L^2(R)$ defined as:

$$(x, y) = \int_{-\infty}^{\infty} \{x(\lambda), y(\lambda)\} d\lambda, \quad x, y \in L^2(R).$$

The formula (1.1) indicates that the mapping

$$\{\xi(t), x\} \leftrightarrow e^{i\lambda t} f_{\lambda}^{1/2} x, \quad x \in R; \quad -\infty < t < \infty, \quad (1.2)$$

is an isometry on $H(\xi)$ into $L^2(R)$. Henceforth, regarding problems such as, for example, that of the structure of the family of subspaces $H_t(\xi)$, $-\infty < t < \infty$ ($H_t(\xi)$ is the closed linear hull of the values $\xi(s)$, $s \leq t$), instead of the stationary process,

$$\{\xi(t), x\}, \quad x \in R; \quad -\infty < t < \infty$$

one can employ its isometric function-space model

$$\{e^{i\lambda t} f_{\lambda}^{1/2} x, \quad -\infty < \lambda < \infty\}, \quad x \in R; \quad -\infty < t < \infty,$$

in the Hilbert space $L^2(R)$. In this sense the family $H_t(\xi)$, $-\infty < t < \infty$, has the same structure as that of the family of subspaces

$$H_t = \overline{\bigvee_{s \leq t} e^{i\lambda s} f_{\lambda}^{1/2} R}, \quad -\infty < t < \infty \quad (1.3)$$

(H_t is the closed linear hull of the subspaces $e^{i\lambda s} f_{\lambda}^{1/2} R \subseteq L^2(R)$, $s \leq t$, etc.). Set

$$H = \bigvee_{-\infty < s < \infty} \overline{e^{i\lambda s} f_{\lambda}^{1/2} R}. \quad (1.4)$$

Denote by P_t the operator of projecting H onto the space H_t and by E_t the unitary operator of multiplying by the function $e^{i\lambda t}$ of λ , $-\infty < \lambda < \infty$.

Obviously,

$$E_u H_t = \overline{\bigvee_{s \leq t} e^{i\lambda(s+u)} f_\lambda^{1/2} R} = H_{t+u} \quad (1.5)$$

for all u, t . Since the operator E_u is unitary, we shall also have $E_u H_t^\perp = H_{t+u}^\perp$ (where $H_t^\perp = H \ominus H_t$ denotes the orthogonal complement of the subspace H_t). Therefore, for any element $x \in H_{t+u}$, we have

$$E_{-u} x \in H_t, \quad P_t E_{-u} x = E_{-u} x, \quad E_u P_t E_{-u} x = x,$$

and for $x \in H_{t+u}^\perp$

$$(t), x\} \leftrightarrow e^{i\lambda t} f_\lambda^{1/2} x, \quad x \in R; \quad -\infty < t < \infty$$

from which it is seen that the product $E_u P_t E_{-u}$ coincides with the projection P_{t+u} . Taking into account that $E_{-u} = E_u^{-1}$, we obtain the following equality:

$$P_{t+u} = E_u P_t E_u^{-1}, \quad -\infty < t < \infty. \quad (1.6)$$

It is actually equivalent to the condition (1.5), since for all t, u

$$P_{t+u} (E_u H_t^\perp) = E_u (P_t H_t^\perp) = 0$$

and

$$E_u H_t^\perp \subseteq H_{t+u}^\perp, \quad H_t^\perp = E_{-u} (E_u H_t^\perp) \subseteq E_{-u} H_{t+u}^\perp,$$

Substituting the parameters t, u for $t+u, -u$, we obtain as well that $H_{t+u}^\perp \subseteq E_u H_t^\perp$, and, consequently,

$$E_u H_t^\perp = H_{t+u}^\perp, \quad E_u H_t = H_{t+u}.$$

The relation (1.6) indicates that the families H_t and H_{t+u} , $-\infty < t < \infty$, are isometric for any u , $-\infty < u < \infty$ (see Sec. 1, Chap. II).

It is clear that if $F_1(t) > F_2(t) > \dots > F_M(t)$ is the ordered system of structural functions for the family of projectors P_t , $-\infty < t < \infty$, then $F_1(t-u) > F_2(t-u) > \dots > F_M(t-u)$ will be an analogous system for the family P_{t+u} , $-\infty < t < \infty$.

Since the families H_t and H_{t+u} , $-\infty < t < \infty$, are isometric, the structural functions $F_j(t)$ and $F_j(t-u)$ must be of the same type:

$$dF_j(t) \sim dF_j(t-u), \quad j=1, \dots, M, \quad (1.7)$$

for all u , $-\infty < u < \infty$.

The property (1.7) is usually referred to as *quasi-invariance* with respect to shifts. If we consider our measures $dF_j(t)$ only on the real line $-\infty < t < \infty$ (without the point $t_0 = -\infty$), then, as is well known, under the condition of quasi-invariance

$$dF_j(t) \sim dt \quad (1.8)$$

(note that the value of $F_j(t_0) = \lim_{t \rightarrow -\infty} F_j(t)$ can be positive).

For the sake of convenience we shall present here a simple proof that any quasi-invariant measure μ is equivalent to a Lebesgue measure. If we take the convolution of a bounded quasi-invariant measure μ with a Gaussian distribution P having the density

$p(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$, then it will turn out that

$$\mu * P(\Delta) = \int \mu(\Delta - t) p(t) dt = 0$$

if and only if $\mu(\Delta) = 0$, i.e., that $\mu * P \sim \mu$. But the convolution $\mu * P$ has a Fourier transform of the form $\varphi(\lambda) e^{-\lambda^2/2}$, where $\varphi(\lambda)$ is the Fourier transform of the bounded measure μ , and, therefore, $\mu * P$ has a smooth density coinciding with the inverse Fourier transform of the function $\varphi(\lambda) e^{-\lambda^2/2}$, $-\infty < \lambda < \infty$. Since the measure $\mu * P$ is also quasi-invariant, its (smooth) density does

not vanish on any interval. Hence it follows that this density is almost everywhere positive.

Recall that we took as the standard structural type of the family of subspaces $U_t = L_t^2(R_M)$, $-\infty < t < \infty$, in a function space $U = L^2(R_M)$, where R_M is some Hilbert space of dimension M ($L_t^2(R_M)$ is the subspace of all the functions $u(s)$, $-\infty < s < \infty$, in $L^2(R_M)$, vanishing for $s > t$).

Let us examine in detail the structural type of the family of subspaces $U_t = L_t^2(R_M)$, $-\infty < t < \infty$. Denote by Q_t the projectors onto U_t . The function $u(s)$, $-\infty < s < \infty$, as an element

of the Hilbert space $U_t = L^2(R_M)$ has $F(t) = \int_{-\infty}^t \|u(s)\|^2 ds$,

$-\infty < t < \infty$, as its structural function. Therefore, any element $u \in U$, such that $u(s) \neq 0$ almost everywhere has a maximal Lebesgue type: $dF(t) \sim dt$. Obviously, the elements

$$u_j(s) = c(s) x_j, \quad -\infty < s < \infty; \quad j = 1, \dots, M$$

(where the numerical function $c(s)$, $-\infty < s < \infty$, is square-integrable and almost everywhere differs from 0, and $x_1, \dots, x_M$ is an orthonormal basis in the Hilbert space R_M) form the maximal system in $L^2(R_M)$ of a maximal structural type (see Sec. 2, Chap. II). In particular, the corresponding multiplicity M coincides with the dimension of the space R_M .

Due to the relation (1.8) the family H_t , $-\infty < t < \infty$, is of the same structural type as that of the family $U_t = L_t^2(R_M)$, $-\infty < t < \infty$, if and only if

$$\bigcap_t H_t = \bigcap_t \left[\bigvee_{s \leq t} e^{i\lambda s} f_{\lambda}^{1/2} R \right] = 0 \quad (1.9)$$

(which is equivalent to the condition $F_j(t_0) = 0$ at the point $t_0 = -\infty$ for all $j = 1, \dots, M$).

Consider now the stationary process $\xi(t)$, $-\infty < t < \infty$ itself. Following the definition of regularity given before (see page 50), we shall call the process $\xi(t)$, $-\infty < t < \infty$, regular if the

family $H_t(\xi)$, $-\infty < t < \infty$, is isometric to the "standard" family $U_t = L_t^2(R_M)$, $-\infty < t < \infty$. We have thus obtained the following result.

Theorem. *The stationary process $\xi(t)$, $-\infty < t < \infty$ is regular if and only if the condition*

$$\bigcap H_t(\xi) = 0. \quad (1.10)$$

is satisfied.

Note that the condition (1.10) served as the basis for a well-known concept of regularity introduced by A. N. Kolmogorov in considering one-dimensional stationary processes ($\dim R = 1$) with discrete time.*

§2. Wold Decomposition and Factorization of the Spectral Density

1. Groups of unitary operators related to a stationary process.

The stationary process $\xi(t)$, $-\infty < t < \infty$, can be pictured as follows: in a Hilbert space $H(\xi)$ there is given a group of unitary operators E_t , $-\infty < t < \infty$, and a certain subspace H^0 whose "shifts" $H^t = E_t H^0$ generate all $H(\xi)$; the aggregate of stationarily-related processes defined by the relations $\{\xi(t), x\} = E_t x$, $-\infty < t < \infty$, where $x \in H^0$, will be a stationary process of the type considered above.

On the other hand, if there are given stationarily-related components $\{\xi(t), x\}$, $-\infty < t < \infty$ (where the parameter $x \in R$, and the Hilbert space R can be arbitrary) then the relation

$$E_t \{\xi(s), x\} = \{\xi(s+t), x\}, \quad -\infty < s < \infty; \quad x \in R, \quad (2.1)$$

*A.N. Kolmogorov, "Stationary Sequences in a Hilbert Space," *Izv. MGU* 2, No. 6 (1941), 1-40.

defines in $H(\xi)$ the unitary operators E_t , $-\infty < t < \infty$, which form a (continuous) group:

$$E_s E_t = E_{s+t}. \quad (2.2)$$

Having taken as H^0 the closure of the subspace of the values $\{\xi(0), x\}$, $x \in R$, we obtain the pair (E_t, H^0) of the type described above. (Earlier, in Sec. 1 we actually exploited a function-space model of a stationary process as the pairs (E_t, H^0) regarding the subspace $H^0 = \overline{f_\lambda^{1/2} R}$ in a function space $L^2(R)$ and the group of unitary operators E_t as operators of multiplication by $e^{i\lambda t}$.)

Suppose that the stationary process $\xi(t)$, $-\infty < t < \infty$, satisfies the condition (1.10) [or, what is the same, its spectral density satisfies the condition (1.9)]. As we know, in this case the family of subspaces $H_t(\xi)$, $-\infty < t < \infty$, is isometric to the family $U_t = L_t^2(R_M)$, $-\infty < t < \infty$, where R_M is a Hilbert space of dimension M (M is the multiplicity of an innovation process for $\xi(t)$, $-\infty < t < \infty$ and the $L_t^2(R_M)$ are subspaces in the function space $L^2(R_M)$, consisting of the functions $u(s)$, $-\infty < s < \infty$, vanishing for $s > t$).

We shall show that there exists an innovation process with uncorrelated increments $X_j(b) - X_j(a)$, $-\infty < a, b < \infty$; $j = 1, \dots, M$, possessing the following property:

$$\begin{aligned} E_t[X_j(b) - X_j(a)] &= X_j(b+t) - X_j(a+t), \\ &-\infty < t < \infty, \end{aligned} \quad (2.3)$$

where E_t is the unitary group on the space $H(\xi)$ related to the stationary process $\xi(t)$ by the relations (2.1). We shall call the innovation process possessing the above property as *stationarily related* to the process $\xi(t)$, $-\infty < t < \infty$.

First we shall show that the group of unitary operations E_t , $-\infty < t < \infty$, is isometric to a group of unitary operators on the function space $U = L^2(R_M)$ having the form $V_t T_t$, where T_t is the shift operator:

$$T_t u(s) = u(s-t), \quad -\infty < s < \infty,$$

and V_t , $-\infty < t < \infty$, is a group of unitary operators on the Hilbert space R_M , acting in $L^2(R_M)$ according to the rule:

$$(V_t u)(s) = V_t u(s), \quad -\infty < s < \infty.$$

Indeed, if we denote by Q_t the projectors on the space U onto the subspace $U_t = L_t^2(R_M)$, $-\infty < t < \infty$, then we shall have:

$$\begin{aligned} Q_{t+s} &= T_s Q_t T_s^{-1}, \quad P_{t+s} = E_s P_t E_s^{-1}, \\ &-\infty < s, t < \infty \end{aligned} \quad (2.4)$$

[compare with (1.5), (1.6)]. Taking into account that

$$P_t = X Q_t X^{-1}, \quad -\infty < t < \infty,$$

where X is some unitary operator on $U = L^2(R_M)$ into $H(\xi)$, from the second relation (2.4) we obtain

$$Q_{t+s} = (X^{-1} E_s X) Q_t (X^{-1} E_s X)^{-1},$$

which is equivalent to the condition

$$(X^{-1} E_s X) U_t = U_{t+s}.$$

Obviously, the unitary operator $(X^{-1} E_s X) T_{-s} = V_s$ has the property that

$$V_s U_t = U_t, \quad -\infty < t < \infty, \quad (2.5)$$

and as we know, only the "operators of multiplying" by a unitary multiplier (on a Hilbert space R_M) possess this property; see the formula (2.21) on page 54. Finally we obtain

$$X^{-1} E_s X = V_s T_s, \quad -\infty < t < \infty.$$

Clearly, for any s', s'' the operators $V_{s'}$ and $T_{s''}$ commute.

Take now the orthonormal basis $x_1, \dots, x_M$ in the Hilbert space R_M and consider the interval-functions on $\Delta = [a, b]$ with values

in a function space $U = L^2(R_M)$, defined by

$$(x_j^b - x_j^a)(t) = V_t x_j \cdot \chi_\Delta(t), \quad -\infty < t < \infty, \quad (2.6)$$

where $\chi_\Delta(t)$ denotes the indication of the interval $\Delta = [a, b]$; $\chi_\Delta(t) = 1$ for $t \in \Delta$, $\chi_\Delta(t) = 0$ for $t \notin \Delta$. Obviously,

$$\begin{aligned} V_s T_s(x_j^b - x_j^a) &= V_s [V_{t-s} x_j \cdot \chi_\Delta(t-s)] = \\ &= V_t x_j \cdot \chi_{\Delta+s}(t) = x_j^{b+s} - x_j^{a+s}. \end{aligned}$$

It is obvious as well that the "increments" $x_j^{b_1} - x_j^{a_1}$, $x_j^{b_2} - x_j^{a_2}$ are orthogonal in $L^2(R_M)$, if the intervals $\Delta_1 = [a_1, b_1]$, $\Delta_2 = [a_2, b_2]$ do not overlap, and for different j they are orthogonal for any Δ_1 and Δ_2 , since for different j , the elements $V_t x_j$ are orthogonal in R_M . Furthermore, since for any s the elements $V_s x_j$, $j = 1, \dots, M$, form an orthonormal system in the Hilbert space $L^2(R_M)$, the functions

$$\begin{aligned} x_j^b - x_j^a &= V_s x_j \cdot \chi_\Delta(s), \\ -\infty < s < \infty, \Delta &= [a, b], -\infty < a; b < \infty; j = 1, \dots, M, \end{aligned}$$

generate the whole subspace $U_t = L_t^2(R_m)$. Set

$$X_j(b) - X_j(a) = X(x_j^b - x_j^a), \quad j = 1, \dots, M, \quad (2.7)$$

where X is the unitary operator on $U = L^2(R_M)$ in $H(\xi)$, satisfying the condition $XU_t = H_t(\xi)$, $-\infty < t < \infty$. It is clear that the process defined above, with the uncorrelated increments $\{X_j(b) - X_j(a)\}_1^M$, $-\infty < a; b < \infty$, is an innovation process for $\xi(t)$, $-\infty < t < \infty$. In this case

$$\begin{aligned} E_t[X_j(b) - X_j(a)] &= E_t X(x_j^b - x_j^a) = \\ &= X[V_t T_t(x_j^b - x_j^a)] = X(x_j^{b+t} - x_j^{a+t}) = \\ &= X_j(b+t) - X_j(a+t), \end{aligned}$$

because by the definition of the operator V_t

$$X^{-1}E_t X = V_t T_t,$$

and, in addition,

$$\|X_j(b) - X_j(a)\|^2 = b - a, \quad j = 1, \dots, M. \quad (2.8)$$

Therefore, the innovation process constructed with the "stochastic differential" $\{dX_j(t)\}_1^M$ is stationarily related to the stationary process $\xi(t)$, $-\infty < t < \infty$.*

Theorem. *The following canonical representation holds:*

$$\{\xi(t), x\} = \int_{-\infty}^t \sum_{j=1}^M c_j(s-t) dX_j(s), \quad -\infty < t < \infty. \quad (2.9)$$

(Usually it is called a *Wold decomposition*.)

Indeed, the value $\{\xi(0), x\} \in H_0(\xi)$ can be represented as

$$\{\xi(0), x\} = \int_{-\infty}^0 \sum_{j=1}^M c_j(s) dX_j(s),$$

where

$$\int_{-\infty}^0 \sum_{j=1}^M |c_j(s)|^2 ds < \infty \quad (2.10)$$

*Another method for constructing an innovation process was suggested by Hanner in a one-dimensional case, $\dim R = 1$ (O. Hanner, Deterministic and non-deterministic stationary random process, Arkiv för Matematik 1, 44 (1950), 161-177. See also G. Kallianpur, V. Mandrekar, Multiplicity and representation theory of purely non-deterministic stochastic processes, Probability Theory and Applications X, 4 (1965), 614-644, where Hanner's method is applied to multidimensional processes).

[see the formula (1.8), Chap. I], and in the case of the innovation process stationarily related to the process $\xi(t)$, we have

$$\begin{aligned}\{\xi(t), x\} &= E_t \{\xi(0), x\} = \\ &= \int_{-\infty}^0 \sum_{j=1}^M c_j(s) dX_j(s+t) = \int_{-\infty}^t \sum_{j=1}^M c_j(s-t) dX_j(s).\end{aligned}$$

2. Factorization of spectral density. As we know, a stationary process with a spectral density $f(\lambda)$ is regular if and only if the condition (1.9) is satisfied, namely,

$$\bigcap_t \left[\overline{\bigvee_{s \leq t} e^{i\lambda s} f_\lambda^{1/2} R} \right] = 0.$$

We shall prove the following assertion.*

Theorem. *For a stationary process to be regular it is necessary and sufficient that the spectral density f_λ permit the factorization*

$$f_\lambda = \varphi_\lambda^* \cdot \varphi_\lambda, \quad -\infty < \lambda < \infty, \quad (2.11)$$

where φ_λ , for almost all λ , is a linear operator on the Hilbert space R into an M -dimensional Hilbert space R_M , and as a function of λ , $-\infty < \lambda < \infty$, $\varphi_\lambda x \in L^2(R)$ for all $x \in R$, with

$$\int_{-\infty}^{\infty} e^{-i\lambda t} \varphi_\lambda x d\lambda = 0 \quad \text{for } t > 0. \quad (2.12)$$

*Apparently, this assertion came first for the case of discrete time in Yu. A. Rozanov's paper "Spectral Theory of Multidimensional Stationary Random Processes with Discrete Time," *Uspekhi Matematicheskikh nauk* **XIII**, 2(80) (1958). At the same time, for a nondegenerate spectral density the problem of factorizing was discussed in N. Wiener, P. Masani, The prediction theory of multivariate stochastic process. II, *Acta Math.*, **99** (1958); H. Helson, D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math.*, **99** (1958).

(Compare with the factorization of the correlation operator B , the operator of multiplying by f_λ considered in Sec. 2, Chap. II.)

PROOF. Consider the almost everywhere defined function

$$u_x(s) = \sum_{j=1}^M c_j(s) x_j, \quad -\infty < s < \infty \quad (u_x(s) = 0 \text{ for } s > 0),$$

where $x_1, \dots, x_M$ is an orthonormalized basis in R_M , and the coefficients $c_1(s), \dots, c_M(s)$ are the same as those in the Wold decomposition (2.9) for the component $\{\xi(t), x\}$ [recall that

$$\int_{-\infty}^{\infty} \sum_{j=1}^M |c_j(s)|^2 ds < \infty, \quad \sum_{j=1}^M |c_j(s)|^2 < \infty$$

for almost all s and the function $u_x(s) \in L^2(R_M)$]. Under the condition (2.8) it follows from the formula (2.9) that

$$E \{ \xi(t_1), x \} \cdot \overline{\{ \xi(t_2), y \}} = \int_{-\infty}^{\infty} \{ u_x(s - t_1), u_y(s - t_2) \} ds.$$

Set

$$\varphi x(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda s} u_x(s) ds, \quad -\infty < \lambda < \infty.$$

It can be readily seen that the correspondence of the elements $\{\xi(t), x\}$, in $H(\xi)$ and $e^{i\lambda t} \varphi x(\lambda)$, $-\infty < \lambda < \infty$, from $L^2(R_M)$ is isometric:

$$\begin{aligned} E \{ \xi(t_1), x \} \cdot \overline{\{ \xi(t_2), y \}} &= \\ &= \int_{-\infty}^{\infty} e^{i\lambda(t_1 - t_2)} \{ f_\lambda x, y \} d\lambda = \int_{-\infty}^{\infty} e^{i\lambda(t_1 - t_2)} \{ \varphi x(\lambda), \varphi y(\lambda) \} d\lambda \end{aligned}$$

for all t_1, t_2 and $x, y \in R$. Hence,

$$\{f_\lambda x, y\} = \{\varphi x(\lambda), \varphi y(\lambda)\}$$

for almost all λ for any $x, y \in R$. The correspondence $x \rightarrow \varphi x(\lambda) \in L^2(R_M)$ [of the same type as $x \rightarrow f_\lambda^{1/2} x \in L^2(R)$] is linear; in other words, φ is the linear operator on the Hilbert space R into the function space $L^2(R_M)$. In this case we can define φ so that on some subspace $\tilde{R}$, everywhere dense in R each $\varphi x(\lambda)$ is a linear operator

$$\varphi x(\lambda) = \varphi_\lambda x, \quad x \in \tilde{R},$$

for almost all λ (one may, for example, take the appropriate $\varphi \tilde{x}_k(\lambda)$, $k = 1, 2, \dots$, and set $\varphi_\lambda x = \sum_{k=1}^n c_k \varphi \tilde{x}_k(\lambda)$ with $x = \sum_{k=1}^n c_k \tilde{x}_k$, where $\tilde{x}_1, \tilde{x}_2, \dots$ is the basis in R). From the equality

$$\{f_\lambda x, y\} = \{\varphi_\lambda x, \varphi_\lambda y\}, \quad x, y \in \tilde{R},$$

it follows that for almost all λ the operator φ_λ in R is bounded and can be extended uniquely from the subspace $\tilde{R}$ to the whole space R (the self-adjoint positive operator f_λ , defined on the whole space R , is bounded for almost all λ). Finally we obtain

$$f_\lambda = \varphi_\lambda^* \cdot \varphi_\lambda,$$

where φ_λ^* is the adjoint of φ_λ with respect to R_M ; φ_λ satisfies the condition (2.12) since

$$\varphi_\lambda x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{i\lambda s} u_x(s) ds, \quad u_x \in L^2(R_M).$$

In turn, the factorization of the spectral density f_λ implies the validity of the regularity condition (1.9)-(1.10). Indeed, taking in the function space $U = L^2(R_M)$ the subspace $U^0 \subseteq L_0^2(R_M)$, generated by the functions

$$u_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda t} \varphi_\lambda x \, d\lambda, \quad x \in R$$

$$(u_x(t) = 0 \quad \text{for } t > 0),$$

we shall have

$$\begin{aligned} E \{ \xi(a), x \} \overline{\{ \xi(b), y \}} &= \int_{-\infty}^{\infty} e^{i\lambda(b-a)} \{ f_\lambda x, y \} \, d\lambda = \\ &= \int_{-\infty}^{\infty} e^{i\lambda(b-a)} \{ \varphi_\lambda x, \varphi_\lambda y \} \, d\lambda = \int_{-\infty}^{\infty} \{ T_a u_x(t), T_b u_y(t) \} \, dt, \end{aligned}$$

where T_s is the shift operator in $L^2(R_M)$. It is seen that the mapping $\{ \xi(a), x \} \leftrightarrow T_a u_x$ is an isometry on $H(\xi)$ into $L^2(R_M)$, and hence the family $H_t(\xi)$, $-\infty < t < \infty$, is isometric to the family of subspaces of the form

$$H_t = \overline{\bigvee_{s \leq t} T_s U^0}, \quad -\infty < t < \infty.$$

Since $H_t \subseteq L_t^2(R_M)$ and $\bigcap_t H_t \subseteq \bigcap_t L_t^2(R_M) = 0$, we have $\bigcap_t H_t(\xi) = 0$.

To conclude we shall introduce the analytical classes $\mathbf{H}^\delta$, $\delta = 1, 2$, of operator functions Γ_λ , $-\infty < \lambda < \infty$, whose values are linear operators on a Hilbert space ($\Gamma_\lambda: R' \rightarrow R''$). We shall say that the function Γ_λ , $-\infty < \lambda < \infty$, belongs to the class $\mathbf{H}^\delta$ if for all $x \in R'$, $y \in R''$ the scalar product $\{ \Gamma_\lambda x, y \}$, as a function of λ , belongs to the space L^δ (on the real line) and

$$\int_{-\infty}^{\infty} e^{-i\lambda t} \{ \Gamma_\lambda x, y \} \, d\lambda = 0 \quad \text{for } t < 0^* \quad (2.13)$$

*For the definition of the classes $\mathbf{H}^\delta$ and their properties see, for example, K. Hoffman's "Banach Spaces of Analytic Functions," Prentice-Hall, 1962,

For example, if the operator function φ_λ satisfies the condition (2.12), the adjoint function $\psi_\lambda = \varphi_\lambda^*$ belongs to the class H^2 since for any $x \in R'$, $y \in R''$ (where $R' = R_M$, $R'' = R$),

$$\{\psi_\lambda x, y\} = \{x, \varphi_\lambda y\} \in L^2 \quad (\text{on the real line})$$

and

$$\int_{-\infty}^{\infty} e^{-i\lambda t} \{\psi_\lambda x, y\} d\lambda = \overline{\int_{-\infty}^{\infty} e^{i\lambda t} \{\varphi_\lambda y, x\} d\lambda} = 0 \quad \text{for } t < 0.$$

Note also that if

$$f_\lambda = \psi_\lambda \cdot \psi_\lambda^*, \quad -\infty < \lambda < \infty,$$

where the operator function $\psi_\lambda \in H^2$, then the function $\varphi_\lambda = \psi_\lambda^*$ will satisfy the condition (2.12).

§3. Multiplicity of a Regular Stationary Process

Consider the regular stationary process $\xi(t)$, $-\infty < t < \infty$, with the components $\{\xi(t), x\}$, $-\infty < t < \infty$, where $x \in R$ (R is a "parametric" Hilbert space). Let f_λ be a spectral density. As we know the type of the family $H_t(\xi)$, $-\infty < t < \infty$, and, in particular, the multiplicity M of an innovation process for $\xi(t)$, $-\infty < t < \infty$, are entirely defined by the spectral density f_λ (see Sec. 1). The question arises of finding the multiplicity M of the innovation process from the spectral density f_λ .

From the factorization condition $f_\lambda = \varphi_\lambda^* \cdot \varphi_\lambda$, where φ_λ is the

and V. I. Krylov's survey "On the functions regular in a half-plane," Math. Sbornik 4, 49 (1938), 9-30. See also: L. L. Privalov's "Boundary Properties of Analytical Functions," Gostekhizdat, 1950; B. Szökefalvi-Nagy and C. Foias, "Harmonic Analysis of Operators in a Hilbert Space," Mir Press, Moscow, 1970.

operator function on the Hilbert space R_M [see the formula (2.11)] it follows that

$$M \geq \dim f_\lambda R. \quad (3.1)$$

Indeed, setting $\psi_\lambda = \varphi_\lambda^*$, we obtain

$$\overline{f_\lambda R} = \overline{\psi_\lambda R_M}, \quad (3.2)$$

since $f_\lambda x = 0$ if and only if $\{f_\lambda x, x\} = \{\varphi_\lambda x, \varphi_\lambda x\} = 0$, and if R_λ^0 is the subspace of all the elements $x \in R$, for which $f_\lambda x = \varphi_\lambda x = 0$, then $\overline{\psi_\lambda R_M} = R \ominus R_\lambda^0 = \overline{f_\lambda R}$.

Note also that the condition (3.2) implies the following property of the spectral density f_λ :

$$\dim f_\lambda R = \text{const.} \quad (3.3)$$

Indeed, if we take any finite-dimensional subspaces $R' \subseteq R_M$, $R'' \subseteq R$ (having chosen from among them the orthonormal bases $x'_1, \dots, x'_m$ and $x''_1, \dots, x''_n$) and consider the projections of the elements $\psi_\lambda x$, $x \in R'$, onto R'' , then it will turn out that the appropriate subspace of elements of the form $\sum_{k=1}^n \{\psi_\lambda x', x''_k\} x''_k$, $x' \in R'$ (denote it by $\varphi_\lambda R'/R''$) has a constant dimension almost everywhere since it coincides with the rank of an analytic matrix with components $\{\psi_\lambda x'_j, x''_k\}$, $j = 1, \dots, m$; $k = 1, \dots, n$, with each minor, as a function of λ , $-\infty < \lambda < \infty$, belonging to one of the well-known analytical classes H^δ with appropriate $\delta > 0$, and is either different from 0 almost everywhere or equal to 0 identically.*

Now it is clear that the dimension

$$\dim \psi_\lambda R' = \max_{R'' \subseteq R} (\dim \varphi_\lambda R'/R'')$$

*See the footnote on pages 80 and 81.

is constant almost everywhere for any finite-dimensional subspace $R' \subseteq R_M$ and, consequently, the dimension

$$\dim \psi_\lambda R_M, \quad \dim \psi_\lambda R_M = \dim f_\lambda R.$$

is constant.

Next let us show that

$$M \leq \dim f_\lambda R. \quad (3.4)$$

Our definition of the multiplicity M of the random process $\xi(t)$, $-\infty < t < \infty$, is related to the family of the projectors P_t onto the subspace $H_t(\xi)$, $-\infty < t < \infty$.

We shall start from the fact that M is equal to the minimal number of cyclic subspaces, whose closed linear hull coincides with the whole space $H(\xi)$ —see Sec. 2, Chap. I—[recall that the “cyclic” subspaces are generated by variables of the type P_t , $-\infty < t < \infty$, where x is some element from $H(\xi)$].

Consider the cyclic subspaces $H_j(\xi)$, $j = 1, \dots, M$ generated by the corresponding increments $X_j(t) - X_j(s)$, $-\infty < s, t < \infty$, of the innovation process in the Wold decomposition (2.9) (a generating element is, for example,

$$\xi_j = \int_{-\infty}^{\infty} c(s) dX_j(s),$$

where the integrand is square-integrable and different from 0 almost everywhere). Choose a function $c(s)$, $-\infty < s < \infty$, such that its shifts $c(s - t)$, $-\infty < s < \infty$, where the parameter t varies within the range $-\infty < t < \infty$, generate the whole space L^2 (on the direct line). It is clear that then the variables

$$\xi_j(t) = \int_{-\infty}^{\infty} c(s - t) dX_j(s), \quad -\infty < t < \infty,$$

generate the whole space $H_j(\xi)$.

The group of unitary operators E_t , $-\infty < t < \infty$, is related to the stationary process $\xi(t)$, $-\infty < t < \infty$, by means of the relation (2.1), and by the well-known Stone representation:*

$$E_t = \int_{-\infty}^{\infty} e^{i\lambda t} d\tilde{E}_\lambda, \quad (3.5)$$

where $\tilde{E}_\lambda$, $-\infty < \lambda < \infty$, is the family of projecting operators in the space $H(\xi)$. Clearly

$$\xi_j(t) = E_t \xi_j, \quad -\infty < t < \infty,$$

from which it follows that the orthogonal subspaces $H_j(\xi)$, $j = 1, \dots, M$, are cyclic also with respect to the family of the projectors $\tilde{E}_\lambda$, $-\infty < \lambda < \infty$ ($H_j(\xi)$ are generated by the variables $\tilde{E}_\lambda \xi_j$, $-\infty < \lambda < \infty$).

In this case all the "cyclic vectors" ξ_j are of the same "Lebesgue type": the functions $\tilde{F}_j(\lambda) = (\tilde{E}_\lambda \xi_j, \xi_j)$ have almost everywhere the positive density

$$f(\lambda) = \left| \int_{-\infty}^{\infty} e^{i\lambda s} c(s) ds \right|^2, \quad -\infty < \lambda < \infty,$$

and, therefore, the family of the projectors $\tilde{E}_\lambda$, $-\infty < \lambda < \infty$, on the space $H(\xi)$ is of the M -multiple "Lebesgue type." Hence, M is equal to the minimal number of cyclic (with respect to the family $\tilde{E}_\lambda$, $-\infty < \lambda < \infty$) subspaces whose closed linear hull coincides with the whole space $H(\xi)$. If we take the function space model (1.2) of the stationary process considered and the corresponding operators E_t , $-\infty < t < \infty$, of multiplication by the function $e^{i\lambda t}$, $-\infty < \lambda < \infty$, in the Hilbert space $L^2(R)$, then it will be easy to specify N of the cyclic subspaces ($N = \dim f_\lambda R$) of the

*See, for example, F. Riesz and B. Sz-Nagy, "Functional Analysis," New York, Frederick Ungar, 1955.

form

$$H_k = \bigvee_{-\infty < t < \infty} e^{i\lambda t} f_\lambda^{1/2} x_k,$$

whose closed linear hull yields the whole space

$$H = \overline{\bigvee_{-\infty < t < \infty} e^{i\lambda t} f_\lambda^{1/2} R};$$

namely, functions $f_\lambda^{1/2} x_k$, $k = 1, \dots, N$ can be taken, whose values form the basis in the subspace $f_\lambda^{1/2} R \subseteq R$ (see the lemma on page 89). From here it follows that $M \leq \dim f_\lambda^{1/2} R (= \dim f_\lambda R)$.

By combining the inequalities (3.1) and (3.4) we obtain the following result.

Theorem. *Under the regularity condition, the multiplicity M of the innovation process is equal to the dimension of the subspace $f_\lambda R \subseteq R$:*

$$M = \dim f_\lambda^{1/2} R \quad (3.6)$$

§4. Regularity Conditions

1. A general criterion of regularity. Our starting point is the theorem of factorization according to which the stationary process $\xi(t)$, $-\infty < t < \infty$, with the spectral density f_λ , $-\infty < \lambda < \infty$ (in a Hilbert space R) is regular if and only if there exists an operator function φ_λ , $-\infty < \lambda < \infty$, of the class $\mathbf{H}^2$, such that

$$f_\lambda = \varphi_\lambda \cdot \varphi_\lambda^* \quad \text{a.e.} \quad (4.1)$$

(the definition of the class $\mathbf{H}^2$ and the analogous class $\mathbf{H}^1$ we shall need further on, was given on page 80). Compared with the condition (2.11) given above, we had φ_λ and φ_λ^* interchanged, so that now φ_λ^* is the linear bounded operator on the Hilbert space R into a M -dimensional Hilbert space R_M . Since always $\dim R \geq M$

($M = \dim f_\lambda R$ a.e.), we can assume $R_M \subseteq R$, and complete the definition of the operator $\varphi_\lambda = (\varphi_\lambda^*)^*$ on the whole space R , setting $\varphi_\lambda(R \ominus R_M) = 0$.

Introduce an operator V_λ^* , setting

$$V_\lambda^*(f_\lambda^{1/2}x) = \varphi_\lambda^*x, \quad x \in R.$$

From the equality (4.1) it is seen that

$$\|V_\lambda^*f_\lambda^{1/2}x\|^2 = \|\varphi_\lambda^*x\|^2, \quad x \in R,$$

and, therefore, V_λ^* maps isometrically the subspace $\overline{f_\lambda^{1/2}R}$ onto the subspace $\overline{\varphi_\lambda^*R}$. Consequently, its adjoint $V_\lambda: V_\lambda(\varphi_\lambda^*x) = f_\lambda^{1/2}x$, $x \in R$, is also isometric; with

$$V_\lambda\overline{\varphi_\lambda^*R} = \overline{f_\lambda^{1/2}R}.$$

Since $\varphi_\lambda^* = V_\lambda^*f_\lambda^{1/2}$, for the adjoint $\varphi_\lambda = (\varphi_\lambda^*)^*$ we obtain that $\varphi_\lambda y = f_\lambda^{1/2}V_\lambda y$ with $y \in \overline{\varphi_\lambda^*R}$, and if we complete the definition of the operator V_λ as to $V_\lambda(R \ominus \overline{\varphi_\lambda^*R}) = 0$, then we shall have

$$\varphi_\lambda = f_\lambda^{1/2}V_\lambda,$$

since the orthogonal complement $R \ominus \overline{\varphi_\lambda^*R}$ of the subspace $\overline{\varphi_\lambda^*R}$ consists of the "zeros" of the operator φ_λ , the adjoint of φ_λ^* . Obviously,

$$\varphi_\lambda R \subseteq f_\lambda^{1/2}R, \quad \overline{f_\lambda^{-1/2}\varphi_\lambda R} = \overline{V_\lambda R} = \overline{f_\lambda^{1/2}R}$$

(here and further $f_\lambda^{-1/2}$ designates the inverse operator to the narrowing $f_\lambda^{1/2}$ on the subspace $\overline{f_\lambda^{1/2}R}$). It is clear that the function

$$\psi_\lambda = \frac{1}{1 - i\lambda} \varphi_\lambda \tag{4.2}$$

satisfies the conditions

$$\psi_\lambda R \subseteq f_\lambda^{1/2}R, \quad \overline{f_\lambda^{-1/2}\psi_\lambda R} = \overline{f_\lambda^{1/2}R},$$

and, in addition,

$$\int_{-\infty}^{\infty} \|\hat{f}_{\lambda}^{-1/2} \psi_{\lambda} x\|^2 d\lambda < \infty, \quad x \in R,$$

since

$$\hat{f}_{\lambda}^{-1/2} \psi_{\lambda} x = \frac{1}{1 - i\lambda} V_{\lambda} x, \quad \|V_{\lambda} x\| \leq \|x\|.$$

In this case the function ψ_{λ} , $-\infty < \lambda < \infty$, belongs to the class $\mathbf{H}^1 \cap \mathbf{H}^2$, because for any $x, y \in R$ the scalar product $\{\psi_{\lambda} x, y\}$ as a function of the class $\mathbf{H}^2$ is the Fourier transform of a certain function $c(t)$ from L^2 (on the real line $-\infty < t < \infty$), $c(t) = 0$ for $t < 0$, and

$$\frac{1}{1 - i\lambda} = \int_0^{\infty} e^{i\lambda t} e^{-t} dt,$$

and, consequently,

$$\{\psi_{\lambda} x, y\} = \frac{1}{1 - i\lambda} \{\varphi_{\lambda} x, y\} = \int_0^{\infty} e^{i\lambda t} c * e^{-t} dt \in \mathbf{H}^1 \cap \mathbf{H}^2,$$

where $c * e^{-t}$, $t > 0$, designates the convolution of those functions.

Let us prove now the following:*

Theorem. *A stationary process with spectral density $\hat{f}_{\lambda}$ is regular if and only if there exists an operator function ψ_{λ} of the class $\mathbf{H}^1$,*

*In the proof we are following the method actually presented in our paper on stationary processes with discrete time. (Yu. A. Rozanov, On Linear Interpolation of Stationary Processes with Discrete Time, DAN SSSR 116 (1959), 923-926); see also Yu. A. Rozanov, Some Approximation Problems in the Theory of Stationary Processes, J. of Multivariate Analysis, 2, 2 (1972), 135-144).

such that

$$\psi_\lambda R \subseteq f_\lambda^{1/2} R, \quad \overline{f_\lambda^{-1/2} \psi_\lambda R} = \overline{f_\lambda^{1/2} R} \quad \text{a.e.} \quad (4.3)$$

and for any $x \in R$ the function $f_\lambda^{-1/2} \psi_\lambda x$, $-\infty < \lambda < \infty$, belongs to the space $L^2(R)$:

$$\int_{-\infty}^{\infty} \|f_\lambda^{-1/2} \psi_\lambda x\|^2 d\lambda < \infty. \quad (4.4)$$

(In this case the existence of the operator function ψ_λ satisfying the conditions (4.3)-(4.4) and belonging to the class $\mathbf{H}^1$ as well as the class $\mathbf{H}^2$ is necessary.)

Above we convinced ourselves that if a stationary process is regular and, therefore, its spectral density f_λ permits the factorization $f_\lambda = \varphi_\lambda \cdot \varphi_\lambda^*$ with the aid of an operator function φ_λ of the class $\mathbf{H}^2$, then the function ψ_λ defined by the formula (4.2) belongs to both the classes $\mathbf{H}^1$, $\mathbf{H}^2$ and satisfies all the conditions of our theorem. Therefore, these conditions are necessary. Let us next prove that they are sufficient.

Recall that we have regularity if and only if the condition (1.9) is satisfied:

$$H_{-\infty} = \bigcap_t \left[\bigvee_{s \leq t} \overline{e^{i\lambda s} f_\lambda^{1/2} R} \right] = 0.$$

Consider in the space

$$H = \overline{\bigvee_{-\infty < s < \infty} e^{i\lambda s} f_\lambda^{1/2} R} \subseteq L^2(R)$$

the orthogonal complement

$$\Delta = H \ominus H_0$$

to the subspace $H_0 = \bigvee_{s \leq 0} \overline{e^{i\lambda s} f_\lambda^{1/2} R}$. Clearly, all the subspaces

$e^{i\lambda t}\Delta$ (consisting of the functions of the form $e^{i\lambda t}x(\lambda)$, $-\infty < \lambda < \infty$, where $x(\lambda)$, $-\infty < \lambda < \infty$, $\in \Delta$) are orthogonal to the subspace $H_{-\infty}$ defined above, since the $e^{i\lambda t}\Delta$ are orthogonal to the corresponding subspaces

$$H_t = \overline{\bigvee_{s \leq t} e^{i\lambda s} f_{\lambda}^{1/2} R} = e^{i\lambda t} H_0.$$

It is clear that if

$$\overline{\bigvee_{-\infty < t < \infty} e^{i\lambda t} \Delta} = H, \quad (4.5)$$

then $H_{-\infty} = 0$.

Define the function-space $\overline{A(\lambda)}$, $-\infty < \lambda < \infty$, for any subspace $A \subseteq L^2(R)$, by choosing in A the complete family of functions $\{a_1(\lambda), a_2(\lambda), \dots\} = S$ and set $\overline{A(\lambda)} = A_S(\lambda)$, where $A_S(\lambda)$ is the closed linear hull in the Hilbert space R of all the values $a_1(\lambda), a_2(\lambda), \dots$. Since for any function $a(\lambda) \in A$ there would be a sequence of linear combinations of the form $\sum_k c_k a_k(\lambda)$, converging in $L^2(R)$ to $a(\lambda)$, some subsequence $\sum_k c_k a_k(\lambda)$ converges to $a(\lambda)$ a.e. to $a(\lambda) \in A_S(\lambda)$ for almost all λ , and hence for any other complete system $S' = \{a'_1(\lambda), a'_2(\lambda), \dots\}$ in $A \subseteq L^2(R)$ we have

$$A_{S'}(\lambda) = A_S(\lambda) \text{ a.e.}$$

In this sense the function space $\overline{A(\lambda)}$, $-\infty < \lambda < \infty$, considered a.e., is defined by the equality $\overline{A(\lambda)} = A_S(\lambda)$ uniquely.

Lemma. *The subspace*

$$L = \overline{\bigvee_{-\infty < t < \infty} e^{i\lambda t} A} \subseteq L^2(R) \quad (4.6)$$

consists of all the functions $x(\lambda)$, $-\infty < \lambda < \infty$, $\in L^2(R)$,

whose values satisfy the condition

$$x(\lambda) \in \overline{A(\lambda)} \quad \text{a.e.} \quad (4.7)$$

PROOF. By the definition of the subspace $L \subseteq L^2(R)$ any function $x(\lambda) \in L$ is the limit of the functions of the type $\sum_k e^{i\lambda t_k} a_k(\lambda)$, where $a_k(\lambda) \in A$, and obviously, the values $x(\lambda)$ satisfy the condition (4.7).

Next we note that for any function $a(\lambda) \in A$ and the scalar measurable function $c(\lambda) \left(\int_{-\infty}^{\infty} |c(\lambda)|^2 \times \|a(\lambda)\|^2 d\lambda < \infty \right)$ the product $x(\lambda) = c(\lambda) a(\lambda)$ is a function from the subspace L , since the function $c(\lambda)$ can be approximated in the mean as closely as desired (with the weight $g(\lambda) = \|a(\lambda)\|^2$) by means of linear combinations of the form $\sum_k c_k e^{i\lambda t_k}$ and for some sequence of the form $\sum_k e^{i\lambda t_k} c_k a(\lambda)$ in $L^2(R)$ we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left\| x(\lambda) - \sum_k e^{i\lambda t_k} c_k a(\lambda) \right\|^2 d\lambda = \\ = \int_{-\infty}^{\infty} \left| c(\lambda) - \sum_k c_k e^{i\lambda t_k} \right|^2 g(\lambda) d\lambda \rightarrow 0. \end{aligned}$$

Analogously, for any functions $a_1(\lambda), \dots, a_r(\lambda) \in A$ and the vector measurable function $c(\lambda) = \{c_1(\lambda), \dots, c_r(\lambda)\}$, satisfying the condition $\int_{-\infty}^{\infty} c(\lambda) g(\lambda) c(\lambda)^* d\lambda < \infty$ (where $g(\lambda)$ is a positive matrix function with the components $g_{pq}(\lambda) = \{a_p(\lambda), a_q(\lambda)\}$, $p, q = 1, \dots, r$), the function $x(\lambda) = \sum_{p=1}^r c_p(\lambda) a_p(\lambda)$ belongs to the subspace L , since the vector function $c(\lambda)$ can be approximated in mean as closely as desired (with the weight $g(\lambda)$),

by means of linear combinations of the form $\sum_k e^{i\lambda t} c_k$ (with the vector coefficients $c_k = \{c_{k1}, \dots, c_{kr}\}$), and for some sequence of functions $\sum_k e^{i\lambda t} \sum_{p=1}^r c_{kp} a_p(\lambda)$ in $L^2(R)$ we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\| x(\lambda) - \sum_k e^{i\lambda t} \sum_{p=1}^r c_{kp} a_p(\lambda) \right\|^2 d\lambda = \\ & = \int_{-\infty}^{\infty} \left[c(\lambda) - \sum_k e^{i\lambda t} c_k \right] g(\lambda) \left[c(\lambda) - \sum_k e^{i\lambda t} c_k \right]^* d\lambda \rightarrow 0. \end{aligned}$$

Take now the arbitrary function $x(\lambda) \in L^2(R)$, satisfying the condition (4.7), and the complete system of functions $a_1(\lambda), a_2(\lambda), \dots, \in A$, whose values generate a subspace $\bar{A}(\lambda) \subseteq R$ for almost all λ . The projection (in a Hilbert space R) of the variable $x(\lambda) \in A(\lambda)$ onto the subspace generated by the finite number of the variables $a_1(\lambda), \dots, a_r(\lambda)$, has the form $x_r(\lambda) = \sum_{p=1}^r c_p(\lambda) a_p(\lambda)$, where the coefficients $c_1(\lambda), \dots, c_r(\lambda)$ as functions of λ are measurable due to the fact that for any functions $x(\lambda), y(\lambda) \in L^2(R)$ the scalar product $\{x(\lambda), y(\lambda)\}$ is a measurable function of λ . We showed above that $x_r(\lambda) \in L$. But for each fixed λ ,

$$\|x(\lambda) - x_r(\lambda)\|^2 \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

with $\|x(\lambda) - x_r(\lambda)\|^2 \leq \|x(\lambda)\|^2$, and, consequently,

$$\int_{-\infty}^{\infty} \|x(\lambda) - x_r(\lambda)\|^2 d\lambda \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

so that $x(\lambda) \in L$. The lemma is proved.

According to this lemma the subspace $H = \overline{\bigvee_{-\infty < t < \infty} e^{i\lambda t} f_{\lambda}^{1/2} R} \subseteq L^2(R)$ consists of all those functions $x(\lambda) \in L^2(R)$ whose values

satisfy the condition

$$x(\lambda) \in \overline{f_\lambda^{1/2} R} \text{ a.e.}, \quad (4.8)$$

and the regularity condition (4.5) can be expressed as

$$\overline{\Delta(\lambda)} = \overline{f_\lambda^{1/2} R} \text{ a.e.}, \quad (4.9)$$

where $\overline{\Delta(\lambda)}$, $-\infty < \lambda < \infty$, designates the function space generated by the subspace $\Delta = H \ominus H_0 \subseteq L^2(R)$.

Suppose now that there exists a certain operator function ψ_λ of the class $\mathbf{H}^1$:

$$\int_{-\infty}^{\infty} e^{-i\lambda t} \{\psi_\lambda x, y\} d\lambda = 0 \text{ for } t < 0$$

for all $x, y \in R$, satisfying the requirements (4.3), (4.4). Clearly, the function $x(\lambda) = f_\lambda^{-1/2} \psi_\lambda x \in L^2(R)$ satisfies the condition (4.8) and, consequently, belongs to the subspace $H = \overline{\bigvee_{-\infty < t < \infty} e^{i\lambda t} f_\lambda^{1/2} R}$.

Furthermore,

$$\int_{-\infty}^{\infty} e^{-i\lambda t} \{x(\lambda), f_\lambda^{1/2} y\} d\lambda = \int_{-\infty}^{\infty} e^{-i\lambda t} \{\psi_\lambda x, y\} d\lambda = 0 \text{ for } t < 0,$$

i.e., the function $x(\lambda)$, $-\infty < \lambda < \infty$, is orthogonal to the subspace $H_0 = \overline{\bigvee_{t < 0} e^{i\lambda t} f_\lambda^{1/2} R}$; in other words, the function $x(\lambda) = f_\lambda^{-1/2} \psi_\lambda x$, $-\infty < \lambda < \infty$, belongs to the subspace $\Delta = H \ominus H_0$. Hence

$$\overline{f_\lambda^{-1/2} \psi_\lambda R} \subseteq \overline{\Delta(\lambda)} \subseteq \overline{f_\lambda^{1/2} R} \text{ a.e.},$$

and due to the relations (4.3) we obtain

$$\overline{\Delta(\lambda)} = \overline{f_\lambda^{1/2} R} \quad \text{a.e.,}$$

from which the regularity condition (4.5) follows.

The theorem is proved completely.

2. Some remarks and examples. Let us examine in more detail the above regularity conditions (4.3)–(4.4).

Consider the finite-dimensional case, $\dim R = N$. Let us suppose that in a unitary space R a certain orthonormal basis $x_1, \dots, x_N$ is chosen and a spectral density f_λ is given by the corresponding matrix $f(\lambda) = \{f_{kj}(\lambda)\}$ with the components $f_{kj}(\lambda) = \{f_\lambda x_k, x_j\}$, $k, j = 1, \dots, N$.

In a one-dimensional case, $\dim R = 1$, the conditions (4.3) and (4.4) obviously imply that the scalar spectral density $f(\lambda)$ must differ from 0 almost everywhere and for some scalar function $\psi(\lambda)$ of class H^1 :

$$\int_{-\infty}^{\infty} |\psi(\lambda)|^2 f(\lambda)^{-1} d\lambda < \infty.$$

This implies that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log [|\psi(\lambda)|^2 f(\lambda)^{-1}]}{1 + \lambda^2} d\lambda = \\ = \int_{-\infty}^{\infty} \frac{\log |\psi(\lambda)|^2}{1 + \lambda^2} d\lambda - \int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1 + \lambda^2} d\lambda < \infty, \end{aligned}$$

and since for the function $\psi(\lambda)$ of class H^1

$$\int_{-\infty}^{\infty} \frac{|\log |\psi(\lambda)|^2|}{1 + \lambda^2} d\lambda < \infty,$$

we arrive at the well-known Kolmogorov-Krein* condition:

*In Kolmogorov's paper cited above (see the footnote on page 72) the regularity condition for one-dimensional stationary processes with discrete

$$\int_{-\infty}^{\infty} \frac{\log f(\lambda)}{1 + \lambda^2} d\lambda > -\infty. \quad (4.10)$$

In the case of an arbitrary finite-dimensional process almost everywhere with invertible spectral density $f(\lambda)$ the conditions (4.3)-(4.4) imply obviously that there exists a non-singular $N \times N$ matrix function $\psi(\lambda) = \{\psi_{kj}(\lambda)\}$ with components from the class H^1 , for which

$$\int_{-\infty}^{\infty} \text{Sp}[\psi(\lambda) f^{-1}(\lambda) \psi(\lambda)^*] d\lambda < \infty.$$

From this it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\log \det [\psi(\lambda) f(\lambda) \psi(\lambda)^*]}{1 + \lambda^2} d\lambda &= \\ &= \int_{-\infty}^{\infty} \frac{\log |\det \psi(\lambda)|^2}{1 + \lambda^2} d\lambda - \int_{-\infty}^{\infty} \frac{\log \det f(\lambda)}{1 + \lambda^2} d\lambda < \infty, \end{aligned}$$

and since for the matrix function $\psi(\lambda)$ of the class H^1

$$\int_{-\infty}^{\infty} \frac{|\log |\det \psi(\lambda)|^2|}{1 + \lambda^2} d\lambda < \infty,$$

we arrive at the well-known regularity condition*

time has been obtained; for a case of continuous time it has been generalized by Krein (M. G. Krein, "On one extrapolation problem due to A. N. Kolmogorov," DAN SSSR, 46 (1944), 306-309).

*For the case of discrete time this condition has been suggested in V. N. Zasukhin's paper, "On the theory of multidimensional stationary processes," DAN SSSR, 33 (1941), 435-437. The case of continuous time has been discussed in Ye. G. Gladyshev's paper "On Multidimensional Stationary Random Processes." Probability Theory and Its Applications, III, 4 (1958), 458-462.

$$\int_{-\infty}^{\infty} \frac{\log \det f(\lambda)}{1 + \lambda^2} d\lambda > -\infty. \quad (4.11)$$

Note that this condition is equivalent to the following:

$$\int_{-\infty}^{\infty} \frac{\log \|f_{\lambda}^{-1}\|^{-1}}{1 + \lambda^2} d\lambda > -\infty,$$

since

$$\log \det f(\lambda) = \sum_{k=1}^N \log m_k(\lambda), \quad \log \|f_{\lambda}^{-1}\|^{-1} = \log m_1(\lambda),$$

where $m_1(\lambda) \leq \dots \leq m_N(\lambda)$ are eigenvalues of the spectral density f_{λ} .

For a regular stationary process the rank of the spectral density $f(\lambda)$ a.e. is equal to constant M (see the theorem in Sec. 3). Consequently, at least one of the principal minors of the order M differs from 0 on a set of positive measure. Suppose it is the determinant of the $M \times M$ -matrix function $f_M(\lambda) = \{f_{k_p k_q}(\lambda)\}$ with elements $f_{k_p k_q}(\lambda) = \{f_{\lambda} x_{k_p}, x_{k_q}\}$, $p, q = 1, \dots, M$. Denote by R_M the subspace in R generated by the elements x_{k_p} , $p = 1, \dots, M$. Clearly, $f_M(\lambda)$ is the (matrix) spectral density of an M -dimensional regular process with components $\{\xi(t), x\}$, $x \in R_M$, and, therefore, $\det f_M(\lambda) \neq 0$ not only on a set of positive measure, but actually almost everywhere. As we have just noted, in the case of the invertible spectral density $f_M(\lambda)$ its least eigenvalue $m(\lambda)$ satisfies the condition

$$\int_{-\infty}^{\infty} \frac{\log m(\lambda)}{1 + \lambda^2} d\lambda > -\infty.$$

In this case, obviously,

$$\begin{aligned}
m(\lambda) &= \inf_{\sum_1^M |c_p|^2 = 1} \sum_{q=1}^M \left| \sum_{p=1}^M f_{k_p k_q}(\lambda) c_p \right|^2 \leq \\
&\leq \inf_{\sum_1^M |c_p|^2 = 1} \sum_{j=1}^N \left| \sum_{p=1}^M f_{j k_p}(\lambda) c_p \right|^2 = \inf_{x \in R_M, \|x\|=1} \|f_\lambda x\|.
\end{aligned}$$

Consider now the subspace $f_\lambda R = R \ominus R_\lambda^0$, where R_λ^0 denotes the null space of the positive operator f_λ^{-1} . Since $\dim f_\lambda R_M = \dim f_\lambda R$, we have $f_\lambda R_M = f_\lambda R$ a.e., which implies that the projection x_λ of any element $x \in R_M$, $x \neq 0$ onto the subspace $f_\lambda R$ differs from 0 a.e. In addition, $f_\lambda x = f_\lambda x_\lambda$ for $x \in R$, and the subspace of all elements x_λ , $x \in R$, coincides with $f_\lambda R$. Clearly,

$$\begin{aligned}
\inf_{x \in R_M, \|x\|=1} \|f_\lambda x\| &\leq \inf_{x \in R_M, \|x_\lambda\|=1} \|f_\lambda x\| = \\
&= \inf_{x \in f_\lambda R, \|x\|=1} \|f_\lambda x\| = \|f_\lambda^{-1}\|^{-1},
\end{aligned}$$

where, as we shall recall, f_λ^{-1} is the inverse of the positive operator f_λ in the unitary space $f_\lambda R$. Therefore, $m(\lambda) \leq \|f_\lambda^{-1}\|^{-1}$ a.e., and, consequently, in the regular case

$$\int_{-\infty}^{\infty} \frac{\log \|f_\lambda^{-1}\|^{-1}}{1 + \lambda^2} d\lambda > -\infty. \quad (4.12)$$

As we know, in addition to this condition it is necessary for regularity that an operator function φ_λ of the class H^2 should exist, having the following property:

$$\varphi_\lambda R \subseteq f_\lambda^{1/2} R, \quad \overline{f_\lambda^{-1/2} \varphi_\lambda R} = \overline{f_\lambda^{1/2} R} \quad \text{a.e.}$$

Clearly, in the finite-dimensional case, $\dim R < \infty$, this is equivalent to the condition

$$f_\lambda R = \varphi_\lambda R \quad \text{a.e.} \quad (4.13)$$

From the general regularity criterion it readily follows that the combination of the conditions (4.12)-(4.13) is not only necessary, but also sufficient for regularity of a finite-dimensional stationary process.*

Indeed, as has been known, under the condition (4.12) the integrable scalar function $\|f_\lambda^{-1}\|^{-1}$ ($\|f_\lambda^{-1}\|^{-1} \leq \|f_\lambda\|$) can be factorized $\|f_\lambda^{-1}\|^{-1} = |\theta(\lambda)|^2$ with the aid of some scalar function of the class H^2 (see the footnote on page 80), and if we take the function $\psi(\lambda) = \theta(\lambda)\varphi(\lambda)$ belonging to the class H^1 , then it will turn out that it satisfies all the conditions (4.3)-(4.4), since

$$\|f_\lambda^{-1/2}\psi_\lambda x\| \leq \|f_\lambda^{-1/2}\| \cdot |\theta(\lambda)| \cdot \|\varphi_\lambda x\| \leq \|\varphi_\lambda x\|, \quad x \in R.$$

It should be noted that the condition (4.12) is equivalent to the following:

$$\int_{-\infty}^{\infty} \frac{\log \det f_M(\lambda)}{1 + \lambda^2} d\lambda > -\infty \quad (4.12')$$

for a certain principal minor of $\det f_M(\lambda)$ of order M of the matrix spectral density $f(\lambda)$ (of rank M). Together with the condition (4.13), it allows us, for example, to obtain readily the following major result: any stationary process having the spectral density $f(\lambda) = \{f_{kj}(\lambda)\}$ with the rational elements $f_{kj}(\lambda)$, $k, j = 1, \dots, N$, is regular.

Let us discuss now some peculiarities of the infinite-dimensional case, $\dim R = \infty$.

Generally speaking, the condition (4.12) need not hold any longer.** For example, if we take a stationary process with a

*In the case of discrete time the analogous conditions are due to H. Helson and D. Louden-slager, Vector-valued Processes, Proc. Fourth Berkeley Symposium Math. Statist. Probability, Berkeley, 1961. A condition of somewhat different type had been found early by R. F. Matveyev, "On Regularity of Multidimensional Stationary Processes." DAN SSSR, 126 (1959).

**A class of spectral densities f_λ (having "a scalar multiple") for which the condition of the type (4.12) is necessary for regularity is indicated in B. Sz-Nagy and C. Foias, "Harmonic analysis of operators in a Hilbert space," 1970.

spectral density of the form $f_{\lambda} x_k = f_k(\lambda) x_k$, $k = 1, 2, \dots$, where $x_1, x_2, \dots$ is an orthonormal basis in R and the scalar spectral densities $f_k(\lambda)$,

$$\int_{-\infty}^{\infty} \frac{\log f_k(\lambda)}{1 + \lambda^2} d\lambda > -\infty,$$

decrease monotonically as

$$\int_{-\infty}^{\infty} \lim_{k \rightarrow \infty} \frac{\log f_k(\lambda)}{1 + \lambda^2} d\lambda = -\infty,$$

then it will turn out that, although the process is regular, the condition (4.12) is violated.

At the same time the condition (4.12) in the case of invertible spectral density is sufficient for regularity.*

Indeed, let f_{λ} be uniformly integrable:

$$\int_{-\infty}^{\infty} \|f(\lambda)\| d\lambda < \infty,$$

*In the case of discrete time this fact has been actually proven in Yu. A. Rozanov's "Spectral Theory of Multidimensional Stationary Random Processes with Discrete Time," *Probability Theory and Its Applications* XIII, 2(80) (1958), 93-142; see also, A. Devinatz, The factorization of operator-valued functions, *Ann. of Math.*, 7B (1961), 458-495. In connection with the condition (4.12) an interesting example given by Lax involving (with the extension to the case of continuous time)

$$\int_{-\infty}^{\infty} \frac{\log f_{\lambda}}{1 + \lambda^2} d\lambda > -cl,$$

where c = some constant, l = the unitary operator in a Hilbert space R , and nevertheless, the corresponding stationary process is not regular (P. Lax, On the Regularity of Spectral Densities, *Probability Theory and Its Applications*, VIII, 3 (1963).

and for almost all λ there exists a bounded inverse operator f_λ^{-1} in R ($R = f_\lambda R$), satisfying the condition

$$\int_{-\infty}^{\infty} \frac{\log \|f_\lambda^{-1}\|^{-1}}{1 + \lambda^2} d\lambda > -\infty.$$

Let

$$\psi_\lambda = \frac{\theta(\lambda)}{1 - i\lambda} I,$$

where $\theta(\lambda)$ is a scalar function of class H^2 factorizing the integrable function $\|f_\lambda^{-1}\|^{-1}$ ($\|f_\lambda^{-1}\|^{-1} \leq \|f_\lambda\|$), and I is the unitary operator in R . Clearly, the operator function ψ_λ of the class $H^1 \cap H^2$ will satisfy the requirements (4.3)–(4.4), and, consequently, the stationary process with the spectral density f_λ will be regular.

In the case where the spectral density f_λ with almost all λ is non-singular on the closed subspace $f_\lambda R$ ($f_\lambda R \subset R$), regularity will occur if in addition to the condition (4.12) we require the function space $\overline{f_\lambda R}$, $-\infty < \lambda < \infty$, to be “analytical” in the sense that

$$f_\lambda R = \overline{\varphi_\lambda R} \quad \text{a.e.} \quad (4.14)$$

[compare with (4.13)] for some operator function φ_λ from the class H^2 .

Indeed, as the function ψ_λ satisfying the regularity conditions (4.3)–(4.4), one may take

$$\psi_\lambda = \theta(\lambda) \cdot \varphi(\lambda),$$

where the scalar function $\theta(\lambda)$ of class H^2 , as before, factorizes the function $\|f_\lambda^{-1}\|^{-1}$ (recall that f_λ^{-1} is the bounded inverse of f_λ on the closed subspace $f_\lambda R = f_\lambda^{1/2} R$ so that $\overline{f_\lambda^{-1/2} \varphi_\lambda R} = \overline{f_\lambda^{1/2} R}$, if $\overline{\varphi_\lambda R} = \overline{f_\lambda R}$).

To conclude we shall deduce a theorem comparing our regularity conditions (4.3)–(4.4).

For brevity we shall call a spectral density regular if the corresponding stationary process is regular.

Suppose that the spectral densities f_λ and g_λ for almost all λ have in the Hilbert space R the same "zeros" and, furthermore, for the elements $x_n \in R$

$$f_\lambda x_n \rightarrow 0 \sim g_\lambda x_n \rightarrow 0. \quad (4.15)$$

If f_λ is regular, and

$$g_\lambda \geq f_\lambda \quad \text{a.e.} \quad (4.16)$$

then the spectral density g_λ is also regular.

Indeed, under the condition (4.15)

$$\overline{f_\lambda^{1/2} R} = \overline{g_\lambda^{1/2} R} \quad (= R_\lambda),$$

with

$$g_\lambda^{1/2} = C_\lambda f_\lambda^{1/2},$$

where C_λ is the invertible operator on the subspace R_λ (from the relation (4.15) it immediately follows that the operators C_λ and C_λ^{-1} defined by the equalities $C_\lambda f_\lambda^{1/2} x = g_\lambda^{1/2} x$ and $C_\lambda^{-1} g_\lambda^{1/2} x = f_\lambda^{1/2} x$, $x \in R$, are continuous in R_λ). Hence, $g_\lambda^{1/2} = f_\lambda^{1/2} C_\lambda^*$ and $f_\lambda^{1/2} = g_\lambda^{1/2} (C_\lambda^*)^{-1}$, from which we obtain that $\overline{g_\lambda^{1/2} R} = \overline{g_\lambda^{1/2} R_\lambda} = \overline{f_\lambda^{1/2} C_\lambda^* R_\lambda} \subseteq \overline{f_\lambda^{1/2} R}$, and, analogously, $\overline{f_\lambda^{1/2} R} \subseteq \overline{g_\lambda^{1/2} R}$, i.e.,

$$\overline{f_\lambda^{1/2} R} = \overline{g_\lambda^{1/2} R} \quad \text{a.e.}$$

For the regular spectral density f_λ there exists an operator function of class $\mathbf{H}^1 \cap \mathbf{H}^2$ satisfying the conditions (4.3)-(4.4). Taking into account that $g_\lambda^{-1/2} = (C_\lambda^*)^{-1} f_\lambda^{-1/2}$, where C_λ^* is the adjoint of C_λ , we arrive at the conclusion that the condition (4.3) is satisfied with respect to the spectral density $g(\lambda)$ as well:

$$\begin{aligned} \psi_\lambda R &\subseteq g_\lambda^{1/2} R \quad (= f_\lambda^{1/2} R), \\ \overline{g_\lambda^{-1/2} \psi_\lambda R} &= \overline{(C_\lambda^*)^{-1} f_\lambda^{-1/2} \psi_\lambda R} = \overline{g_\lambda^{1/2} R}. \end{aligned}$$

Furthermore, since $g_\lambda \geq f_\lambda$, we have $g_\lambda^{-1} \leq f_\lambda^{-1}$ a.e., and, consequently, with respect to $g(\lambda)$ the condition (4.4) is satisfied as well:

$$\int_{-\infty}^{\infty} \|g_\lambda^{-1/2} \psi_\lambda x\|^2 d\lambda \leq \int_{-\infty}^{\infty} \|f_\lambda^{-1/2} \psi_\lambda x\|^2 d\lambda < \infty$$

for all $x \in R$. Therefore, the spectral density $g(\lambda)$ is regular.

In particular, in the finite-dimensional case it follows from the above that if the non-singular spectral density f_λ is regular, then any spectral density g_λ , $g_\lambda \geq f_\lambda$, will be also regular. For the infinite-dimensional case with constraints of the type (4.15), this is, generally speaking, false. (Namely, for any function $x(\lambda) \in H^0$, $x(\lambda) \neq 0$, where H^0 is the closure in $L^2(R)$ of the subspace of all the functions $g_\lambda^{1/2} x$, $-\infty < \lambda < \infty$, $x \in R$, in the case of the regularity of g_λ the one-dimensional "stationary process" $\{e^{i\lambda t}, x(\lambda)\}$ is also regular and the condition

$$\int_{-\infty}^{\infty} \frac{\log \|x(\lambda)\|}{1 + \lambda^2} d\lambda > -\infty;$$

needs to be satisfied; in particular, $x(\lambda) \neq 0$ a.e. If we take strictly positive (constant) operators P and Q in R , such that $Q \geq P$ and $Qx_n \rightarrow x_0 \neq 0$, $Px_n \rightarrow 0$ for some sequence $x_n \in R$, take a scalar function $\theta(\lambda)$ of the class $H^1 \cap H^2$, $|\theta(\lambda)| \leq 1$, then it will turn out that the operator function $f_\lambda = |\theta(\lambda)|^2 P^2$ is regular, $f_\lambda = \varphi_\lambda \cdot \varphi_\lambda^*$, $\varphi_\lambda = \theta(\lambda) P$, and the function g_λ :

$$g_\lambda = \begin{cases} |\theta(\lambda)|^2 P^2 & \text{for } |\lambda| \leq 1, \\ |\theta(\lambda)|^2 Q^2 & \text{for } |\lambda| > 1, \end{cases}$$

though it majorizes f_λ , will not be regular, since

$$g_\lambda^{1/2} x_n \rightarrow x(\lambda) = \begin{cases} 0 & \text{for } |\lambda| \leq 1, \\ |\theta(\lambda)| x_0 & \text{for } |\lambda| > 1. \end{cases}$$

Let us make more precise the fact that this counterexample is essentially infinite-dimensional. As P and Q the following operators can be taken: Q is the positive invertible operator for which $QR \neq R$, and $P = Q\pi Q$, where π is the operator of projecting onto the orthogonal complement to the element $x_0 \in R$, $x_0 \notin QR$.*

Having in mind the regularity conditions (4.3)-(4.4), we note that for the densities f_λ and g_λ , $|\lambda| > 1$, defined above $\overline{g_\lambda^{-1/2} f_\lambda R} = \overline{Q^{-1}(Q\pi Q)R} = \pi R \neq \overline{g_\lambda^{1/2} R} = R$, and this is evidence of the impossibility of selecting the operator function $\psi_\lambda \in H^1 \cap H^2$, satisfying the conditions (4.3) with respect to both f_λ and g_λ (as was done in proving our comparison theorem).

*See R. G. Douglas, On factoring positive operator functions, J. Math. Mech., 16 (1966), 119-126.

Chapter IV

EQUIVALENT RANDOM PROCESSES

§1. The Concept of Equivalence. Probability Interpretation in the Case of Gaussian Distributions

Let $\xi(t) = \{\xi(t), x\}_{x \in R}$ and $\eta(t) = \{\eta(t), x\}_{x \in R}$ be two random processes on an interval $t_0 < t < T$, where the parameter x runs through some set R . Having in mind the question under which conditions their innovation processes are of the same type, we shall consider the mapping

$$A: \{\eta(t), x\} \rightarrow \{\xi(t), x\}, \quad x \in R; \quad t_0 < t < T. \quad (1.1)$$

We shall say that the process $\xi(t) = \{\xi(t), x\}_{x \in R}$ is equivalent to the process $\eta(t) = \{\eta(t), x\}_{x \in R}$ on the interval $t_0 < t < T$, if this mapping extends to a linear bounded invertible operator A on the Hilbert space $H(\eta)$ into the Hilbert space $H(\xi)$ and, in addition, if the difference $I - A^*A$ is a Hilbert-Schmidt operator (recall that $H(\eta)$ and $H(\xi)$ are the closed linear hulls of the corresponding variables $\{\eta(t), x\}$ and $\{\xi(t), x\}$, $x \in R$; $t_0 < t < T$).

As was suggested before (see Sec. 2, Chap. II), under the condition that A is a bounded linear operator, instead of the primary processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, one can consider the generalized process

$$\xi(u) = Au, \quad u \in U, \quad (1.2)$$

on a Hilbert space $U = H(\eta)$ with the distinguished family of subspaces $U_t = H_t(\eta)$, $t_0 < t < T$.

The question of uniqueness of type of the innovation processes for $\xi(t)$ and $\eta(t)$ (on the interval $t_0 < t < T$) is actually the question: under what conditions will the family of subspaces

$$H_t(\xi) = \overline{AH_t(\eta)}, \quad t_0 < t < T, \quad (1.3)$$

be isometric to the family $H_t(\eta)$, $t_0 < t < T$:

$$H_t(\xi) = XH_t(\eta), \quad t_0 < t < T, \quad (1.4)$$

for some isometric operator X ?

It was proved (see the theorem in Sec. 3, Chap. II), that if the correlation operator $B = A^*A$ is invertible, and, in addition, the difference $I - B$ is a Hilbert-Schmidt operator, then the families $H_t(\xi)$ and $H_t(\eta)$, $t_0 < t < T$, will be isometric, and, consequently, the innovation processes for primary random processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, will be of the same type. Therefore, the equivalent processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, have innovation processes of the same type.

Note that since $A = XB^{1/2}$, where X is a unitary operator, and

$$\begin{aligned} I - (A^{-1})^*(A^{-1}) &= I - XBX^{-1} = \\ &= X(I - B)X^{-1} = X(I - A^*A)X^{-1}, \end{aligned}$$

then in the case where the difference $I - A^*A$ is a Hilbert-Schmidt operator, the difference $I - (A^{-1})^*A^{-1}$ will be an operator of the same type. Therefore, the definition of the equivalence $\xi(t) \sim \eta(t)$ is symmetric. It can be easily seen that it is also transitive and, of course, reflexive.

The equivalence condition introduced above is well known for Gaussian random processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, when it implies the equivalence (mutual absolute continuity) of their probability distributions P_ξ and P_η on any function space.*

*This equivalence condition of Gaussian distributions is due to J. Feldman, "Equivalence and perpendicularity of Gaussian processes," *Pacif. J. Math.*, 8

It should be mentioned here that the question under our consideration about the innovation process structure involves only those properties of random processes which can be completely defined by means of their second moments (correlation function), so that these processes can be considered Gaussian without loss of generality. Thus, the following statement holds.

Theorem. *Equivalent Gaussian random processes have innovation processes of the same type.*

§2. Equivalence of Stationary Processes

1. Problem Statement. *Hilbert-Schmidt, operators on a function space $L^2(R)$.* Suppose that $\xi(t) = \{\xi(t), x\}_{x \in R}$ is a stationary process with the components $\{\xi(t), x\}$, $x \in R$ (R is a parametric Hilbert space), which we shall consider on a certain interval $t_0 < t < T$.

Denote by f_λ its spectral density. Recall that f_λ is a positive operator function on a Hilbert space R , such that, for all $x, y \in R$,

$$\mathbf{E} \{ \xi(t), x \} \cdot \overline{\{ \xi(s), y \}} = \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \{ f_\lambda x, y \} d\lambda.$$

Along with the spectral density f_λ we shall use another property of the stationary process $\xi(t)$ —its correlation function

$$B_\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} f_\lambda d\lambda, \quad -\infty < t < \infty,$$

being the Fourier (weak) transform of the (weakly) integrable operator function f_λ , $-\infty < \lambda < \infty$.

(1958), 699–708. The detailed discussion of various problems related to the equivalence of Gaussian distribution can be found in Yu. A. Rozanov, "Infinite-dimensional Gaussian Distribution," American Mathematical Society, 1971.

Suppose that $\eta(t) = \{\eta(t), x\}_{x \in R}$ is another stationary process with spectral density g_λ and correlation function $B_\eta(t)$ in a parametric Hilbert space R . The question is under which conditions imposed on the spectral densities f_λ and g_λ [or on the correlation functions $B_\xi(t)$ and $B_\eta(t)$] the processes $\xi(t)$ and $\eta(t)$ will be equivalent on the interval $t_0 < t < T$.

Having in mind the isometry

$$\xi(t) \leftrightarrow e^{i\lambda t} f_\lambda^{1/2}, \quad \eta(t) \leftrightarrow e^{i\lambda t} g_\lambda^{1/2}$$

(see Sec. 1, Chap. III), we shall consider in the Hilbert space $L^2(R)$ the subspaces $H(f)$ and $H(g)$ each of which is the closure of the corresponding linear subspaces of all functions of the form $f_\lambda^{1/2}x(\lambda)$ and $g_\lambda^{1/2}x(\lambda)$, where

$$x(\lambda) = \sum_{t_0 < t_k < T} e^{i\lambda t_k} x_k, \quad -\infty < \lambda < \infty; \quad (2.1)$$

namely,

$$H(f) = \overline{\bigvee_{t_0 < t < T} e^{i\lambda t} f_\lambda^{1/2} R}, \quad H(g) = \overline{\bigvee_{t_0 < t < T} e^{i\lambda t} g_\lambda^{1/2} R}. \quad (2.2)$$

The equivalence of the processes $\xi(t)$ and $\eta(t)$ on the interval $t_0 < t < T$ implies that the mapping

$$A: g_\lambda^{1/2}x(\lambda) \rightarrow f_\lambda^{1/2}x(\lambda) \quad (2.3)$$

is a linear bounded invertible operator on the Hilbert space $H(g)$ into the Hilbert space $H(f)$, and the difference $I - A^*A$ is Hilbert-Schmidt.*

*Note that the methods and the results presented below can be easily extended to the operator functions f_λ and g_λ of "slow growth"

$$\int_{-\infty}^{\infty} |\lambda|^{-2n} \|f_\lambda\| d\lambda < \infty, \quad \int_{-\infty}^{\infty} |\lambda|^{-2n} \|g_\lambda\| d\lambda < \infty$$

Before we speak about various equivalence conditions, we shall demonstrate that any Hilbert-Schmidt operator in the function space $L^2(R)$ in some sense is "a kernel"—this fact is well known in the finite-dimensional case $\dim R < \infty$.

Let us now consider the Hilbert space $S_2(R)$ of all Hilbert-Schmidt operators in the Hilbert space R having defined the scalar product $\varphi, \psi \in S_2(R)$ as

$$\begin{aligned} \{\varphi, \psi\} &= \text{Sp } \psi^* \varphi = \sum_k \{\varphi e_k, \psi e_k\} = \\ &= \sum_{k, j} \{\varphi e_k, e_j\} \overline{\{\psi e_k, e_j\}}, \end{aligned} \quad (2.4)$$

where $e_1, e_2, \dots$ is an orthonormal basis in R . It is well known that the sums on the right-hand side of the equality (2.4) do not depend on the choice of the orthonormal basis in R and that the space $S_2(R)$ is complete with respect to the "trace norm":

$$\|\varphi\|^2 = \text{Sp } \varphi^* \varphi = \sum_k \|\varphi e_k\|^2 = \sum_{k, j} |\{\varphi e_k, e_j\}|^2.$$

Consider the space of all measurable operator functions $\varphi(\lambda, \mu)$, $-\infty < \lambda, \mu < \infty$, in the Hilbert space R satisfying the condition

$$\varphi(\lambda, \mu) \in S_2(R) \text{ and } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(\lambda, \mu)|^2 d\lambda d\mu < \infty, \quad (2.5)$$

for any $n > 0$, which corresponds to the case of generalized stationary processes and processes with stationary increments. Instead of the primary functions $x(\lambda)$ of the type (2.1), we need to take only the functions

$$x(\lambda) = \int_{t_0}^T e^{i\lambda t} c(t) dt,$$

which are the Fourier transforms of sufficiently smooth functions $c(t)$, $-\infty < t < \infty$, vanishing outside of the interval (t_0, T) .

having defined the scalar product with the aid of the formula

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\varphi(\lambda, \mu), \psi(\lambda, \mu)\} d\lambda d\mu. \quad (2.6)$$

This will be a complete Hilbert space; denote it by $L^2(R \times R)$.

An example of the operator function $\varphi(\lambda, \mu) \in L^2(R \times R)$ is the function

$$\varphi(\lambda, \mu) = x_0(\lambda) \times y_0(\mu), \quad -\infty < \lambda, \mu < \infty,$$

defined by

$$\varphi(\lambda, \mu)x = x_0(\lambda) \cdot \{y_0(\mu), x\}, \quad x \in R, \quad (2.7)$$

where $x_0(\lambda), y_0(\lambda) \in L^2(R)$. Indeed,

$$\|\varphi(\lambda, \mu)\|^2 = \|x_0(\lambda)\|^2 \sum_k |\{y_0(\mu), e_k\}|^2 = \|x_0(\lambda)\|^2 \cdot \|y_0(\mu)\|^2$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \|\varphi(\lambda, \mu)\|^2 d\lambda d\mu = \int_{-\infty}^{\infty} \|x_0(\lambda)\|^2 d\lambda \cdot \int_{-\infty}^{\infty} \|y_0(\mu)\|^2 d\mu.$$

If $x_1(\lambda), x_2(\lambda), \dots$ is some orthonormal system in the function space $L^2(R)$, then the system of operator functions

$$\varphi_{kj}(\lambda, \mu) = x_k(\lambda) \times x_j(\mu), \quad k, j = 1, 2, \dots, \quad (2.8)$$

will also be orthonormal in the space $L^2(R \times R)$, since

$$\begin{aligned} \{\varphi_{ij}(\lambda, \mu), \varphi_{pq}(\lambda, \mu)\} &= \\ &= \{x_i(\lambda), x_p(\lambda)\} \sum_k \{x_j(\mu), e_k\} \cdot \overline{\{x_q(\mu), e_k\}} = \\ &= \{x_i(\lambda), x_p(\lambda)\} \cdot \{x_j(\mu), x_q(\mu)\} \end{aligned}$$

and

$$\begin{aligned} (\varphi_{ij}, \varphi_{pq}) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\varphi_{ij}(\lambda, \mu), \varphi_{pq}(\lambda, \mu)\} d\lambda d\mu = \\ &= \int_{-\infty}^{\infty} \{x_i(\lambda), x_p(\lambda)\} d\lambda \cdot \int_{-\infty}^{\infty} \{x_j(\mu), x_q(\mu)\} d\mu = 0 \end{aligned}$$

with $i \neq p$ and $j \neq q$.

Moreover, taking advantage of the fact that any Hilbert-Schmidt operator is the limit in trace norm of "finite-dimensional" operators,* it can be shown that if the orthonormalized system of functions $x_1(\lambda), x_2(\lambda), \dots$ is complete in $L^2(R)$, then the corresponding system of the operator functions $\varphi_{kj}(\lambda, \mu) = x_k(\lambda) \times x_j(\mu)$, $k, j = 1, 2, \dots$, will be a complete orthonormalized system in the space $L^2(R \times R)$ (we shall not need this statement and therefore we shall omit its proof).

Theorem. Any Hilbert-Schmidt operator on a function space $L^2(R)$ is defined by a kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$.

PROOF. Let F be a Hilbert-Schmidt operator [on a certain subspace $H \subseteq L^2(R)$]. This means that

$$\sum_{k,j} |(Fx_k, x_j)|^2 < \infty \quad (2.9)$$

for any complete orthonormalized system of functions $x_k = x_k(\lambda)$, $k = 1, 2, \dots$, in the space $H \subseteq L^2(R)$. Take the orthonormalized system of the operator functions

$$\varphi_{kj}(\lambda, \mu) = x_k(\lambda) \times x_j(\mu), \quad k, j = 1, 2, \dots$$

and set

$$\varphi(\lambda, \mu) = \sum_{k,j} b_{kj} \varphi_{kj}(\lambda, \mu),$$

*See, for instance, N. Dunford and J. T. Schwartz, "Linear Operators. Spectral Theory," John Wiley and Sons, 1971.

where the coefficients are

$$b_{kj} = (Fx_k, x_j), \quad \sum_{k,j} |b_{kj}|^2 < \infty.$$

It is clear that $\varphi(\lambda, \mu)$ is an element of the (complete) function space $L^2(R \times R)$. For all k, j and p

$$\varphi_{kj}(\lambda, \mu) x_p(\mu) = x_k(\lambda) \cdot \{x_j(\mu), x_p(\mu)\} \in H$$

and

$$\int_{-\infty}^{\infty} \varphi_{kj}(\lambda, \mu) x_p(\mu) d\mu = \begin{cases} x_k(\lambda) & \text{if } j = p, \\ 0 & \text{if } j \neq p. \end{cases}$$

It can be readily seen that the operator function $\varphi(\lambda, \mu)$ possesses the following property: for any function $x(\lambda) = \sum_p a_p x_p(\lambda)$ $\left(\sum_p |a_p|^2 < \infty\right)$ in the space $H \subseteq L^2(R)$

$$\int_{-\infty}^{\infty} \varphi(x, \mu) x(\mu) d\mu = \sum_k c_k x_k(\lambda),$$

where

$$c_k = \sum_p a_p b_{pk}, \quad \sum_k |c_k|^2 \leq \sum_p |a_p|^2 \cdot \sum_{k,j} |b_{kj}|^2 < \infty.$$

One can say that the operator Φ defined on the space $L^2(R)$ by

$$\Phi x(\lambda) = \sum_k c_k x_k(\lambda) \quad \left(x(\lambda) = \sum_j a_j x_j(\lambda)\right),$$

is prescribed by the kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$, and this

operator Φ coincides with the primary operator F , since

$$\begin{aligned} Fx(\lambda) &= \sum_j a_j \cdot Fx_j(\lambda) = \sum_j a_j \sum_k (Fx_j, x_k) x_k(\lambda) = \\ &= \sum_k \left(\sum_j a_j b_{jk} \right) x_k(\lambda) = \sum_k c_k x_k(\lambda). \end{aligned}$$

Thus, we have demonstrated that any operator F of the Hilbert-Schmidt type on a function space $H \subseteq L^2(R)$ is prescribed by a certain kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$, namely,

$$Fx(\lambda) = \int_{-\infty}^{\infty} \varphi(\lambda, \mu) x(\mu) d\mu. \quad (2.10)$$

In turn, as can be easily seen, any kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$ prescribes a certain operator F of the Hilbert-Schmidt type on a space $L^2(R)$, with

$$\|F\|^2 = \text{Sp } F^*F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(\lambda, \mu)|^2 d\lambda d\mu. \quad (2.11)$$

Further on we shall need some other properties of the Fourier transform of operator functions $\varphi(\lambda, \mu) \in L^2(R \times R)$, defined by

$$b(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda s - \mu t)} \varphi(\lambda, \mu) d\lambda d\mu, \quad (2.12)$$

$$-\infty < s; t < \infty.$$

In the case of the integrable function $|\varphi(\lambda, \mu)|$ this is a usual integral of a function with values in the separable Hilbert space $S_2(R)$. In the general case the transform (2.12) can be defined in the sense of weak convergence. In fact, if we choose some orthonormal basis $e_1, e_2, \dots$ in the Hilbert space R and consider the usual

Fourier transform of the square-integrable functions $\{\varphi(\lambda, \mu) e_k, e_j\}$, denoting it by $\{b(s, t) e_k, e_j\}$, it will turn out that for any linear combinations $x = \sum c'_k e_k$, $y = \sum c''_j e_j$ the function

$$\{b(s, t) x, y\} = \sum_{k, j} c'_k c''_j \{b(s, t) e_k, e_j\}$$

coincides with the Fourier transform $\{\varphi(\lambda, \mu) x, y\}$, and hence we can consider this transform as a bilinear functional on the subspace $\tilde{R}$ everywhere dense in R of all linear combinations of the form $x = \sum c_k e_k$. In this case

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k, j} |\{b(s, t) e_k, e_j\}|^2 ds dt &= \\ &= \sum_{k, j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\{b(s, t) e_k, e_j\}|^2 ds dt = \\ &= 4\pi^2 \sum_{k, j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\{\varphi(\lambda, \mu) e_k, e_j\}|^2 d\lambda d\mu = \\ &= 4\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k, j} |\{\varphi(\lambda, \mu) e_k, e_j\}|^2 d\lambda d\mu = \\ &= 4\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(\lambda, \mu)|^2 d\lambda d\mu < \infty, \end{aligned}$$

from which it is seen that

$$\sum_{k, j} |\{b(s, t) e_k, e_j\}|^2 < \infty \text{ a.e.,}$$

and, consequently, the bilinear functional $\{b(s, t) x, y\}$ is given by a certain operator $b(s, t)$ of the Hilbert-Schmidt type (on the space R). As an operator function $b(s, t)$ belongs to the function space $L^2(R \times R)$:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |b(s, t)|^2 ds dt = 4\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\varphi(\lambda, \mu)|^2 d\lambda d\mu < \infty. \quad (2.13)$$

We call the operator function $b(s, t)$, $-\infty < s, t < \infty$, the Fourier transform of $\varphi(\lambda, \mu) \in L^2(R \times R)$.

In turn, the operator function $\varphi(\lambda, \mu)$ can be obtained from the operator function $b(s, t) \in L^2(R \times R)$ with the aid of the inverse Fourier transformation:

$$\varphi(\lambda, \mu) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda s - \mu t)} b(s, t) ds dt, \quad (2.14)$$

$$-\infty < \lambda, \mu < \infty.$$

By using the well-known properties of the Fourier transform of scalar functions one can easily convince oneself, for example, that if for some integer n

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + \lambda^2)^n (1 + \mu^2)^n |\varphi(\lambda, \mu)|^2 d\lambda d\mu < \infty, \quad (2.15)$$

the operator function $b(s, t)$ has all (weak) derivatives up to the order $n - 1$ in each variable, with

$$\begin{aligned} \frac{\partial^{k+m}}{\partial s^k \partial t^m} b(s, t) &= \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-i\lambda)^k (i\mu)^m e^{-i(\lambda s - \mu t)} \varphi(\lambda, \mu) d\lambda d\mu, \end{aligned}$$

and, moreover, the function $c(s, t) = \frac{\partial^{2(n-1)}}{\partial s^{n-1} \partial t^{n-1}} b(s, t)$ is absolutely continuous with respect to the Lebesgue measure $ds \times dt$

in the sense that the additive set function

$$m(\Delta) = c(s_2, t_2) - c(s_2, t_1) - c(s_1, t_2) + c(s_1, t_1), \\ \Delta = (s_1, s_2] \times (t_1, t_2],$$

can be represented in the form of the integral

$$m(\Delta) = \int_{s_1}^{s_2} \int_{t_1}^{t_2} \left[\frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) \right] ds dt,$$

$\frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t)$ being the Fourier transform of the operator function $\psi(\lambda, \mu) = (-i\lambda^n)(i\mu)^n \varphi(\lambda, \mu) \in L^2(R \times R)$ and, consequently, satisfying the condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) \right|^2 ds dt < \infty. \quad (2.16)$$

On the other hand, if the operator function $b(s, t)$, let us say, has compact support: $b(s, t) = 0$ outside of some square $t_0 < s$; $t < T$, and has the derivative $\frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t)$ satisfying the condition (2.16), i.e., $\frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) \in L^2(R \times R)$, then all lower-order derivatives [including the function $b(s, t)$] also belong to the space $L^2(R \times R)$. For instance,

$$\int_{t_0}^T \int_{t_0}^T \left| \frac{\partial^{2k}}{\partial s^k \partial t^k} b(s, t) \right|^2 ds dt \leqslant \\ \leqslant (T - t_0)^{4(n-k)} \int_{t_0}^T \int_{t_0}^T \left| \frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) \right|^2 ds dt.$$

In this case the inverse Fourier transforms of the operator functions $\frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t)$ and $b(s, t)$ are related by the equality

$$\psi(\lambda, \mu) = (-i\lambda)^n (i\mu)^n \varphi(\lambda, \mu),$$

obtained, as for scalar functions, with the aid of repeated integration and integration by parts of the identity

$$\begin{aligned} (i\lambda)^{-n} (-i\mu)^{-n} \psi(\lambda, \mu) &= \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(i\lambda)^{-n} (-i\mu)^{-n} e^{i(\lambda s - \mu t)}] \frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) ds dt. \end{aligned}$$

Therefore, the operator function $\varphi(\lambda, \mu)$ defined by the formula (2.14), satisfies the condition

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda|^{2n} |\mu|^{2n} |\varphi(\lambda, \mu)|^2 d\lambda d\mu &= \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(\lambda, \mu)|^2 d\lambda d\mu < \infty, \end{aligned} \quad (2.15')$$

and, therefore, the condition (2.15).

To conclude we shall present one more simple fact concerning the operator functions $\varphi(\lambda, \mu) \in L^2(R \times R)$. Namely, suppose that the measurable operator functions α_λ and β_λ , $-\infty < \lambda < \infty$, in the space R satisfy the condition

$$\int_{-\infty}^{\infty} \|\alpha_\lambda\|^2 d\lambda < \infty, \quad \int_{-\infty}^{\infty} \|\beta_\lambda\|^2 d\lambda < \infty.$$

Then the product $\beta_\lambda^* \varphi(\lambda, \mu) \alpha_\mu$ also belongs to the space $L^2(R \times R)$ and, for all $x, y \in R$,

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \varphi(\lambda, \mu) \alpha_{\mu} x d\mu, \beta_{\lambda} y \right\} d\lambda = \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\beta_{\lambda}^* \varphi(\lambda, \mu) \alpha_{\mu} x, y\} d\lambda d\mu. \quad (2.17)$$

2. General equivalence conditions. For the stationary processes $\xi(t)$ and $\eta(t)$ with the spectral densities f_{λ} and g_{λ} (in the Hilbert space R) the equivalence on the interval $t_0 < t < T$ implies that the operator defined by the relation (2.3) on the space $H(g)$ into the space $H(f)$ is bounded and invertible, and the difference $I - A^*A$ is Hilbert-Schmidt on $H(g)$. More precisely, define the operator A by:

$$Au(\lambda) = f_{\lambda}^{1/2} x(\lambda), \quad -\infty < \lambda < \infty$$

on all the functions

$$u(\lambda) = g_{\lambda}^{1/2} x(\lambda) \\ \left(x(\lambda) = \sum_{t < t_k < T} e^{i\lambda t_k} x_k, \quad -\infty < \lambda < \infty \right),$$

the spaces $H(g)$ and $H(f)$ coinciding with the closure in $L^2(R)$ of the subspaces of the corresponding functions $g_{\lambda}^{1/2} x(\lambda)$ and $f_{\lambda}^{1/2} x(\lambda)$ [see (2.1)-(2.3)].

We shall assume from now on that the spectral density $g(\lambda)$ is uniformly integrable:

$$\int_{-\infty}^{\infty} \|g(\lambda)\| d\lambda < \infty. \quad (2.18)$$

Assume $B = A^*A$ and $F = I - B$. It is clear that the condition of boundedness and invertibility of the operator H is equivalent to the fact that the operator B possesses the same properties. Next, if $F = I - B$ is a Hilbert-Schmidt operator, then $B = I - F$ will be a bounded operator, and the condition of invertibility of the positive

operator $B = A^* A$:

$$\inf_{\|u\|=1} (Bu, u) = 1 - \sup_{\|u\|=1} (Fu, u) > 0$$

is equivalent to the fact that the self-adjoint operator F (of Hilbert-Schmidt type) has no eigenvalues equal to 1—its maximal eigenvalue is less than 1:

$$\mu = \sup_{\|u\|=1} (Fu, u) < 1. \quad (2.19)$$

It should be mentioned here that the boundedness and invertibility of the primary operator A implies the following*

$$\int_{-\infty}^{\infty} \{f_{\lambda} x(\lambda), x(\lambda)\} d\lambda \asymp \int_{-\infty}^{\infty} \{g(\lambda) x(\lambda), x(\lambda)\} d\lambda, \quad (2.20)$$

where $x(\lambda)$ runs through all the functions of the form $x(\lambda) = \sum_{t_k < t < T} e^{i\lambda t_k} x_k$.

As we showed above (see the theorem from Sec. 1), the Hilbert-Schmidt operator on the subspace $H(g) \subseteq L^2(R)$ is defined by a certain kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$, which with respect to the operator $F = I - B$ is equivalent to the following: for the complete [in $H(g)$] system of functions

$$u(\lambda) = e^{i\lambda t} g_{\lambda}^{1/2} x, \quad t_0 < t < T, \quad x \in R,$$

and a certain kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$, we have:

*Here and further the notation $\alpha \asymp \beta$ for the variables α, β implies that

$$0 < c_1 \leq \frac{\alpha}{\beta} \leq c_2 < \infty,$$

where c_1, c_2 are constants.

$$\begin{aligned}
 (Fu, v) &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{i\mu t} \varphi(\lambda, \mu) g_{\mu}^{1/2} x d\mu, e^{i\lambda s} g_{\lambda}^{1/2} y \right\} d\lambda = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda s - \mu t)} \{g_{\lambda}^{1/2} \varphi(\lambda, \mu) g_{\mu}^{1/2} x, y\} d\lambda d\mu
 \end{aligned}$$

[see the formula (2.17)]. On the other hand, for $u(\lambda) = e^{i\lambda s} g_{\lambda}^{1/2} x$, $v(\lambda) = e^{i\lambda t} g_{\lambda}^{1/2} y$ we have

$$\begin{aligned}
 (Fu, v) &= (u, v) - (Bu, v) = \\
 &= \int_{-\infty}^{\infty} e^{i\lambda(s-t)} \{g_{\lambda} x, y\} d\lambda - \int_{-\infty}^{\infty} e^{i\lambda(s-t)} \{f_{\lambda} x, y\} d\lambda = \\
 &= \int_{-\infty}^{\infty} e^{i\lambda(s-t)} \{(g_{\lambda} - f_{\lambda}) x, y\} d\lambda = b(s-t),
 \end{aligned}$$

where $b(s-t)$ is the difference of the correlation functions:

$$b(s-t) = B_{\eta}(s-t) - B_{\xi}(s-t), \quad t_0 < s, t < T.$$

Comparing the expressions presented above, we arrive at the following result.

Lemma. *The difference $I - B$ is a Hilbert-Schmidt operator if and only if the difference of the correlation functions permits the representation*

$$\begin{aligned}
 b(s-t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda s - \mu t)} g_{\lambda}^{1/2} \varphi(\lambda, \mu) g_{\mu}^{1/2} d\lambda d\mu, \quad (2.21) \\
 &\quad t_0 < t, s < T,
 \end{aligned}$$

with the aid of a certain kernel $\varphi(\lambda, \mu) \in L^2(R \times R)$.

Here the integrand

$$\psi(\lambda, \mu) = g_{\lambda}^{1/2} \varphi(\lambda, \mu) g_{\mu}^{1/2}$$

belongs to the space $L^2(R \times R)$, since $\int_{-\infty}^{\infty} \|g_{\lambda}^{1/2}\|^2 d\lambda < \infty$ [see the condition (2.18)], and the formula (2.21) indicates that the operator function $b(s-t)$, $t_0 < s$, $t < T$, can be extended to a certain function

$$b(s, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\lambda s - \mu t)} \psi(\lambda, \mu) d\lambda d\mu, \\ -\infty < s, t < \infty,$$

whose Fourier transform

$$\psi(\lambda, \mu) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\lambda s - \mu t)} b(s, t) ds dt, \\ -\infty < \lambda, \mu < \infty,$$

satisfies the condition

$$g_{\lambda}^{-1/2} \psi(\lambda, \mu) g_{\mu}^{-1/2} \in L^2(R \times R). \quad (2.22)$$

It is also obvious that if the function $b(s-t)$, $t_0 < s$, $t < T$, permits such a continuation, then it also permits a representation of the type (2.21) with the kernel $\varphi(\lambda, \mu) = g_{\lambda}^{-1/2} \psi(\lambda, \mu) g_{\mu}^{-1/2}$.

Therefore, the following assertion* holds.

Theorem. *The difference $I - B$ is a Hilbert-Schmidt operator if and only if the difference of the correlation functions $b(s-t)$, $t_0 < s$, $t < T$, permits the continuation into a certain function $b(s, t)$, $-\infty < s, t < \infty$, on $L^2(R \times R)$ with the Fourier transform $\psi(\lambda, \mu)$ satisfying the condition (2.22).*

*This theorem and the following ones are an extension of the results for one-dimensional processes. The detailed proof of them is presented in I. A. Ibragimov and Yu. A. Rozanov (Gaussian Random Processes), Moscow, Nauka, 1970.

However strange it looks at first sight, in the case of the finite interval (t_0, T) with rather wide bounds on the spectral densities f_λ and g_λ , from the condition that the difference $I - B$ is a Hilbert-Schmidt operator (and even simply a self-adjoint completely continuous operator), the invertibility of the operator B follows, and in this case the theorem presented above yields, as we shall see later, a rather effective criterion of equivalence.

Suppose that the spectral density f_λ is non-singular on at least some set of positive measure. Suppose that the spectral density g_λ is regular, more precisely,

$$\int_{-\infty}^{\infty} \frac{\log \|g_\lambda^{-1}\|^{-1}}{1 + \lambda^2} d\lambda > -\infty \quad (2.23)$$

[compare with the condition (4.12) in Chap. III]. Then the following result holds.

Theorem. *Under the condition that the difference $I - B$ is compact, the operator $B = A^*A$ will be invertible.*

The proof of this assumption is based on the fact that under the condition (2.23) any element $u(\lambda) \in H(g)$ is representable as

$$u(\lambda) = g_\lambda^{1/2} x(\lambda), \quad -\infty < \lambda < \infty, \quad (2.24)$$

where $x(\lambda)$ is the entire analytic function being the meansquare limit with the weight $w(\lambda) = \|g_\lambda^{-1}\|^{-1}$ of analytic functions of the type (2.1):

$$x(\lambda) = \lim_{t_0 < t_k < T} \sum e^{i\lambda t_k} x_k.$$

Indeed, any element $u(\lambda) \in H(g)$ is, by the definition of the space $H(g)$, the limit in $L^2(R)$ of some sequence $u_n(\lambda) = g_\lambda^{1/2} x_n(\lambda)$, where $x_n(\lambda)$ are analytic functions of the type (2.1). The sequence $u_n(\lambda)$ is fundamental in the space $L^2(R)$, so that

$$\begin{aligned} \int_{-\infty}^{\infty} \|g_\lambda^{1/2} (x_n(\lambda) - x_m(\lambda))\|^2 d\lambda &\geq \\ &\geq \int_{-\infty}^{\infty} \|x_n(\lambda) - x_m(\lambda)\|^2 w(\lambda) d\lambda \rightarrow 0 \quad \text{as } n, m \rightarrow \infty, \end{aligned}$$

since for any $x \in R$

$$\|x\|^2 w(\lambda) = \left\| \frac{g_\lambda^{-1/2}}{\|g_\lambda^{-1/2}\|} g_\lambda^{1/2} x \right\|^2 \leq \|g_\lambda^{1/2} x\|^2,$$

and, therefore, there exists a function $x(\lambda)$, $\int_{-\infty}^{\infty} \|x(\lambda)\|^2 \times w(\lambda) d\lambda < \infty$, such that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \|x_n(\lambda) - x(\lambda)\|^2 w(\lambda) d\lambda = 0.$$

Therefore, for some subsequence we have

$$x_n(\lambda) \rightarrow x(\lambda), \quad g_\lambda^{1/2} x_n(\lambda) \rightarrow g_\lambda^{1/2} x(\lambda) \quad \text{a.e.},$$

which leads us to the formula (2.24). Since for any $x \in R$

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |\{x_n(\lambda), x\} - \{x(\lambda), x\}|^2 w(\lambda) d\lambda = 0,$$

the scalar function $\{x(\lambda), x\}$, being the limit of functions of the form

$$\{x_n(\lambda), x\} = \sum_{t_0 < t_k < T} e^{i\lambda t_k} c_k$$

is an entire analytic function.* For instance, under the condition

*The detailed proof of this statement is presented in the book cited on page 35. The description of the analytic functions $x(\lambda)$ in the scalar case is given in M. G. Krein's, "On the fundamental approximation problem of the theory of stationary random processes extrapolation," DAN SSSR (1954), 13-16; see also: N. Levinson, H. McKean, "Weighted trigonometrical approxi-

$$\|g_\lambda^{-1}\|^{-1} \geq K(1 + \lambda^2)^{-n}, \quad (2.25)$$

where n is some positive integer [which is stronger than the regularity condition (2.23)] for the appropriate functions $x(\lambda)$ the following representation can be given:*

$$x(\lambda) = \sum_{k=0}^{n-1} (i\lambda)^k e^{i\lambda t_0} x_k + (1 + i\lambda)^n \int_0^T e^{i\lambda t} c(t) dt,$$

where the function $c(t)$ belongs to the space $L^2(R)$.

Thus, let the space $H(g)$ consist of elements of the form $u(\lambda) = g_\lambda^{1/2} x(\lambda)$, where $x(\lambda)$ are entire analytic functions.

Assume that the operator $B = A^*A$ is not invertible. Then, as we showed above [see (2.19)], the self-adjoint compact operator $F = I - B$ must have an eigenvalue equal to 1. Denote by $u(\lambda) \in H(g)$ its corresponding eigenfunction. Take the sequence of analytic functions $x_n(\lambda)$ of the type (2.1) such that $u(\lambda) = g_\lambda^{1/2} x(\lambda)$ is the limit of the elements $u_n(\lambda) = g_\lambda^{1/2} x_n(\lambda)$ and $x(\lambda) = \lim x_n(\lambda)$ a.e. It is seen that $f_\lambda^{1/2} x_n(\lambda) \rightarrow f_\lambda^{1/2} x(\lambda)$ a.e. and simultaneously the sequence of elements $f_\lambda^{1/2} x_n(\lambda) = Au_n(\lambda)$ is such that

$$(Au_n, Au_n) = (Bu_n, u_n) = (u_n, u_n) - (Fu_n, u_n) \rightarrow 0,$$

since $u_n \rightarrow u$, $Fu_n \rightarrow u$. Therefore, in $L^2(R)$ the sequence $f_\lambda^{1/2} x_n(\lambda)$ converges to zero, so that

$$f_\lambda^{1/2} x(\lambda) = 0 \quad \text{a.e.,}$$

mation with application to the germ field of stationary Gaussian noise," Acta Math, 112, 1-2 (1964), 99-143.

*Note that this representation can be used for finding the projectors Q_t onto the subspaces

$$H_t(g) = \overline{\bigvee_{t_0 < s < t} e^{i\lambda s} g_\lambda^{1/2} R}, \quad t_0 < t < T$$

(see the problem of prediction on a finite interval in Yu. A. Rozanov, "Stationary Random Processes," Moscow, Fizmatgiz, 1963).

and, consequently, $x(\lambda) = 0$ on a set of positive measure (where $f_\lambda^{1/2}$ is not singular), and for the analytic function $x(\lambda)$ this is possible only in the case where $x(\lambda) \equiv 0$. But this contradicts the fact that $u(\lambda) = g_\lambda^{1/2} x(\lambda)$ is an eigenfunction with a single eigenvalue, and, therefore, the operator $B = A^* A$ is invertible.

3. Some conclusions and examples. Let us show that the general conditions given in Sec. 2 of the equivalence of the stationary processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, with the spectral densities f_λ and g_λ in a Hilbert space R are sufficiently effective.

For short we shall call the spectral densities f_λ and g_λ equivalent if the corresponding processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$ are equivalent.

Assume that

$$g_\lambda \leq K(1 + \lambda^2)^{-n} I. \quad (2.26)$$

Then, as follows from the condition (2.22) (see the theorem on page 119), for equivalence it is necessary that the difference of the correlation functions

$$b(s-t) = B_\eta(s-t) - B_\xi(s-t) = \int_{-\infty}^{\infty} e^{i\lambda(s-t)} (g_\lambda - f_\lambda) d\lambda$$

on the square $t_0 < s, t < T$ coincide with a certain operator function $b(s, t)$, $-\infty < s, t < \infty$, in $L^2(R \times R)$ with the Fourier transform $\varphi(\lambda, \mu)$, satisfying the condition (2.15):

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + \lambda^2)^n (1 + \mu^2)^n |\varphi(\lambda, \mu)|^2 d\lambda d\mu < \infty,$$

since

$$|g_\lambda^{-1/2} \varphi(\lambda, \mu) g_\mu^{-1/2}|^2 \geq K^{-2} (1 + \lambda^2)^n (1 + \mu^2)^n |\varphi(\lambda, \mu)|^2.$$

Indeed, in the case considered

$$\|x\|^2 = \|g_\lambda^{1/2} g_\lambda^{-1/2} x\|^2 \leq K(1 + \lambda^2)^{-n} \|g_\lambda^{-1/2} x\|^2$$

and

$$\begin{aligned} |g_\lambda^{-1/2} \varphi(\lambda, \mu) g_\mu^{-1/2}|^2 &= \sum_p \|g_\lambda^{-1/2} \varphi(\lambda, \mu) g_\mu^{-1/2} x_p\|^2 \geq \\ &\geq \sum_p K^{-1} (1 + \lambda^2)^n \|\varphi(\lambda, \mu) g_\mu^{-1/2} x_p\|^2 = \\ &= K^{-1} (1 + \lambda^2)^n |\varphi(\lambda, \mu) g_\mu^{-1/2}|^2 = \\ &= K^{-1} (1 + \lambda^2)^n |g_\mu^{-1/2} \varphi(\lambda, \mu)^*|^2 = \\ &= K^{-1} (1 + \lambda^2)^n \sum_p \|g_\mu^{-1/2} \varphi(\lambda, \mu)^* x_p\|^2 \geq \\ &\geq K^{-2} (1 + \lambda^2)^n (1 + \mu^2)^n \sum_p \|\varphi(\lambda, \mu)^* x_p\|^2 = \\ &= K^{-2} (1 + \lambda^2) (1 + \mu^2)^n |\varphi(\lambda, \mu)^*|^2 = \\ &= K^{-2} (1 + \lambda^2)^n (1 + \mu^2)^n |\varphi(\lambda, \mu)|^2. \end{aligned}$$

Since $b(s, t) = b(s - t)$ with $t_0 < s, t < T$, it follows from the condition (2.15) that for the equivalent spectral densities f_λ and g_λ the operator function $b(t)$ on the interval $(-\tau, \tau)$, where $\tau = T - t_0$, must have all derivatives up to the order $2n$, the $2n$ -th derivative $b^{(2n)}(t)$, $-\tau < t < \tau$, satisfying the condition

$$\int_{t_0}^T \int_{t_0}^T |b^{(2n)}(s - t)|^2 ds dt < \infty. \quad (2.27)$$

Assume now, that the relation (2.25) is satisfied, in other words,

$$g_\lambda \geq K(1 + \lambda^2)^{-n} I. \quad (2.28)$$

If the operator function $b(t)$, $t_0 < t < T$, possesses the properties described above and has a derivative $\frac{\partial^{2n}}{\partial t^{2n}} b(t)$, satisfying the

condition (2.27), then the operator function $b(s, t) = b(s - t)$ can be continued from the square $t_0 < s, t < T$ onto the whole plane $-\infty < s, t < \infty$ into a certain finite smooth function $b(s, t)$, such that*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \frac{\partial^{2n}}{\partial s^n \partial t^n} b(s, t) \right|^2 ds dt < \infty.$$

The Fourier transform $\psi(\lambda, \mu)$ of this function $b(s, t)$ will satisfy the conditions (2.15') and (2.15), and, consequently, the condition (2.22)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| g_{\lambda}^{-1/2} \psi(\lambda, \mu) g_{\mu}^{-1/2} \right|^2 d\lambda d\mu < \infty,$$

since in our case

$$\begin{aligned} \left| g_{\lambda}^{-1/2} \psi(\lambda, \mu) g_{\mu}^{-1/2} \right|^2 &\leq \|g_{\lambda}^{-1/2}\| \cdot \|g_{\mu}^{-1/2}\| \cdot |\psi(\lambda, \mu)|^2 \leq \\ &\leq K^2 (1 + \lambda^2)^n (1 + \mu^2)^n |\psi(\lambda, \mu)|^2. \end{aligned}$$

Hence the difference $I - A^*A$ will be a Hilbert-Schmidt operator which under the condition (2.28) is the same as the equivalence of the spectral densities g_{λ} and f_{λ} (assuming f_{λ} is nonsingular a.e.).

*In the orthonormal basis $e_1, e_2, \dots \in R$ the operator function $b(s, t)$ with values in $S_2(R)$ is given by a certain matrix $b_{kj}(s, t)$, $\sum_{k, j} |b_{kj}(s, t)|^2 < \infty$ and the question is about an obvious continuation of

each scalar component $b_{kj}(s, t)$ from the square $t_0 < s, t < T$ onto the whole plane $-\infty < s, t < \infty$, preserving the smoothness and integrability properties of the type

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k, j} |b_{kj}(s, t)|^2 ds dt < \infty.$$

Therefore, if the difference of the correlation functions on the interval $(-\tau, \tau)$, $\tau = T - t_0$, has the derivative $b^{(2n)}(t)$, satisfying the condition (2.27), then the spectral densities f_λ and g_λ will be equivalent.

In particular, the spectral densities f_λ and g_λ will be equivalent no matter what n is in the condition (2.28) and what the finite interval (t_0, T) is, if

$$f_\lambda = g_\lambda + \Delta_\lambda,$$

where Δ_λ is the arbitrary finite function with the square-integrable trace norm:

$$\Delta_\lambda = 0 \quad \text{for } |\lambda| > a, \quad \int_{-a}^a |\Delta_\lambda|^2 d\lambda < \infty. \quad (2.29)$$

Indeed, in this case the operator function

$$b(t) = - \int_{-\infty}^{\infty} e^{i\lambda t} \Delta_\lambda d\lambda, \quad t_0 < s, \quad t < T,$$

will have derivatives of all orders satisfying the condition (2.27).

In the finite-dimensional case, $\dim R < \infty$, we obtain, for example, the result that if the spectral density f_λ differs from the spectral density g_λ on any finite interval (in particular, it is possible that $f_\lambda \equiv 0$ with $|\lambda| \leq a$), then f_λ and g_λ are equivalent.

Note also that for the spectral density g_λ of the type

$$k(1 + \lambda^2)^{-n} I \leq g_\lambda \leq K(1 + \lambda^2)^{-n} I$$

instead of the condition (2.29) one can require that

$$\int_{-\infty}^{\infty} |g_\lambda^{-1/2} \Delta_\lambda g_\lambda^{-1/2}|^2 d\lambda < \infty. \quad (2.29')$$

We shall formulate our principal result as a separate theorem.

Theorem. *If $g_\lambda \leq K(1 + \lambda^2)^{-n}$, then for the equivalence of f_λ and g_λ on the interval $(t_0, t_0 + \tau)$ it is necessary that the operator function*

$$b(t) = B_\eta(t) - B_\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} (g_\lambda - f_\lambda) d\lambda$$

on the interval $-\tau < t < \tau$ has all derivatives up to the order $2n$, with

$$\int_0^\tau \int_0^\tau |b^{(2n)}(s-t)|^2 ds dt < \infty.$$

If $g_\lambda \geq k(1 + \lambda^2)^{-n}I$, then the conditions indicated are sufficient for equivalence.

To conclude, as a consequence of the results obtained above, we shall describe an innovation process for the class of stationary processes $\xi(t)$, $t_0 < t < T$, with the correlation function $B(t)$ of the following type: on the open interval $(0, \tau)$, $\tau = T - t_0$, the function $B(t)$ has all derivatives up to the order $2n$, the function $B^{(2n-1)}(t)$ being continuous everywhere on the interval $(-\tau, \tau)$, except at the point $t=0$, where it has a discontinuity of the first kind, and the function $B^{(2n)}(t)$ satisfies the condition

$$\int_0^\tau \int_0^\tau |B^{(2n)}(t-s)|^2 ds dt < \infty.$$

Each process $\xi(t)$ is equivalent to the stationary process $\eta(t)$, $t_0 < t < T$, with spectral density of the type

$$g(\lambda) = \frac{1}{|P(i\lambda)|^2},$$

where $P(i\lambda)$ is a polynomial of degree n with the corresponding

coefficient of highest degree $(t\lambda)^n$. Consequently, the innovation process $X(t) = \{X_k(t)\}_{k=1}^{n+1}$, $t_0 < t < T$, for $\xi(t)$ will be of the same type as that for the process $\eta(t)$; namely, the first n components will be constant:

$$X_k(t) \equiv \xi^{(k-1)}(t_0), \quad k = 1, \dots, n;$$

they generate the "primary" subspace*

$$\bigcap_{h>0} H_{t_0+h}(\xi),$$

and one component $X_{n+1}(t)$ describing the future innovation of the subspaces $H_t(\xi)$, $t_0 < t < T$, is a process of the Brownian motion type (see Sec. 3, Chap. I).

§3. Random Processes Equivalent to a Wiener Process

From the viewpoint of our approach to the evolution of the subspaces $H_t(\xi)$, $t_0 < t < T$, related to the corresponding random process $\xi(t)$, $t_0 < t < T$, the Wiener process $\eta(t)$, $t_0 < t < T$ (with the components $\{\eta(t), x\}$, $t_0 < t < T$, where the parameter x runs through some Hilbert space R) can be taken as a certain standard. Namely, the natural isometry

$$\{\eta(t), x\} \leftrightarrow \chi_t(s) x \quad (3.1)$$

between the variables $\{\eta(t), x\} \in H(\eta)$ and the elements $\chi_t(s) x \in L^2(R)$ allows us to identify $H_t(\eta)$ with the subspaces $L_t^2(R)$, $t_0 < t < T$; here the scalar function $\chi_t(s)$, $t_0 < t < T$ is the indicator of the interval $(t_0, t]$, $L^2(R)$ is the Hilbert space of all measurable R -valued functions $u(s)$, $t_0 < s < T$, square-integrable,

*Under the conditions indicated above the process $\xi(t)$ has exactly $n - 1$ derivatives.

with the scalar product

$$(u, v) = \int_{t_0}^T \{u(s), v(s)\} ds, \quad u, v \in L^2(R),$$

and $L_t^2(R)$ is the subspace of all functions $u(s) \in L^2(R)$, such that $u(s) = 0$ with $s > t$. We assume that the standard Wiener process $\eta(t)$, $t_0 < t < T$, has the normalized correlation function

$$B_\eta(s, t) = \min(s - s_0, t - t_0) \cdot I, \quad t_0 < s, t < T. \quad (3.2)$$

Let $\xi(t)$, $t_0 < t < T$, be some random process with the components $\{\xi(t), x\}$, $t_0 < t < T$, $x \in R$, and let $B_\xi(s, t)$ be its correlation (operator) function in a Hilbert space R :

$$E \{ \xi(s), x \} \cdot \overline{\{ \xi(t), y \}} = \{ B_\xi(s, t) x, y \}, \quad x, y \in R. \quad (3.3)$$

As we know, the family of subspaces $H_t(\xi)$, $t_0 < t < T$, will be isometric to the family $L_t^2(R)$, $t_0 < t < T$, if, let us say, the process $\xi(t)$, $t_0 < t < T$, is equivalent to the Wiener process $\eta(t)$, $t_0 < t < T$. In this case the equivalence implies that the operator A :

$$A: \chi_t(s) x \rightarrow \{ \xi(t), x \}, \quad (3.4)$$

defined on the whole $[L^2(R)]$ system of elements

$$u(s) = \chi_t(s) x \quad (t_0 < s < T; \quad x \in R; \quad t_0 < t < T) \quad (3.5)$$

can be continued up to a linear bounded invertible operator A , such that the difference $I - A^*A$ is a Hilbert-Schmidt operator on the function space $L^2(R)$.

According to the general theorem from Subsec. 1, Sec. 2 any Hilbert-Schmidt operator in the function space $L^2(R)$ is defined by a certain kernel $K(s, t) \in L^2(R \times R)$ —by the operator function in the Hilbert space R with the values $K(s, t) \in S_2(R)$ and with square-integrable trace norm:

$$\int_{t_0}^T \int_{t_0}^T |K(s, t)|^2 ds dt < \infty, \quad (3.6)$$

where

$$|K(s, t)|^2 = \text{Sp}[K(s, t)^* \cdot K(s, t)].$$

Consequently, under the equivalence condition

$$(u, v) - (Au, Av) = \int_{t_0}^T \int_{t_0}^T \{K(s, t) u(s), v(t)\} ds dt, \quad (3.7)$$

where $K(s, t)$ is some kernel in $L^2(R \times R)$.

Obviously, the relation (3.7) will be satisfied for all the elements $u, v \in L^2(R)$, if it is satisfied for some complete system of elements [let us say for the system (3.5)]. With $u(s) = \chi_{t_1}(s)x$, $v(t) = \chi_{t_2}(t)y$ the relation (3.7) can be represented in an operator form:

$$B_\eta(t_1, t_2) - B_\xi(t_1, t_2) = \int_{t_0}^{t_1} \int_{t_0}^{t_2} K(s, t) ds dt. \quad (3.8)$$

It is also obvious that if the representation (3.8) holds, then the corresponding operator $I - A^*A$ will be a Hilbert-Schmidt operator.

Next, if the difference $I - B$ ($B = A^*A$) is a Hilbert-Schmidt operator, then the condition of invertibility of the bounded operator B can be expressed as follows: the operator $I - B$ has no eigenvalue equal to 1 (for this see page 116).

Finally we have arrived at the following result.

Theorem. *For equivalence of the random processes $\xi(t)$ and $\eta(t)$, $t_0 < t < T$, it is necessary and sufficient that the difference $b(s, t) = B_\eta(s, t) - B_\xi(s, t)$ be absolutely continuous with respect to $ds \times dt$, more precisely,*

$$b(s, t) = \int_{t_0}^s \int_{t_0}^t K(s', t') ds' dt', \quad t_0 < s, t < T, \quad (3.9)$$

where the derivative $K(s, t) = \frac{\partial^2}{\partial s \partial t} b(s, t)$ has the square-integrable trace norm:

$$\int_{t_0}^T \int_{t_0}^T \left| \frac{\partial^2}{\partial s \partial t} b(s, t) \right|^2 ds dt < \infty, \quad (3.10)$$

and, in addition, the kernel $K(s, t)$ as an operator in $L^2(R)$ has no eigenvalue equal to one.*

This criterion of equivalence holds in the case of a more complex "standard" process $\eta(t)$, $t_0 < t < T$, as well, having a correlation function of the form $B_\eta(s, t) = \min(F(s), F(t)) \cdot I$ where $F(t)$, $t_0 < t < T$, is a left-continuous monotone non-decreasing scalar function. One needs only to take instead of the Lebesgue measure $ds \times dt$ the corresponding measure $dF(s) \times dF(t)$.

Example. Let $\eta(t)$, $t_0 < t < T$, be a one-dimensional random process with uncorrelated increments and $F(t) = E |\eta(t)|^2$, $t_0 < t < T$, be its structural function. Let $a(t)$, $t_0 < t < T$, be a random process, such that

$$\int_{t_0}^T E |a(t)|^2 dF(t) < \infty.$$

Let, finally,

$$\xi(t) = \int_{t_0}^t a(s) dF(s) + \eta(t), \quad t_0 < t < T. \quad (3.11)$$

*Apparently, the first publications on the conditions of such a kind for the scalar case are due to L. H. Shepp, "Radon-Nikodym derivatives of Gaussian measures," Ann. Math. Stat., 37, 2 (1966), 321-354. Note that these conditions are readily obtained from the conditions of equivalence to a white-noise stationary process with the simple spectral density $g(\lambda) = I$.

Remembering that the projection of the variable $a(u)$ onto the subspace $H_t(\eta)$ is representable as the stochastic integral

$\int_{t_0}^t c(u, v) d\eta(v)$, where the integrand satisfies the condition

$$\int_{t_0}^t |c(u, v)|^2 dF(v) \leq \mathbb{E} |a(u)|^2,$$

we obtain

$$\begin{aligned} \mathbb{E} a(u) \overline{\eta(t)} &= \\ &= \mathbb{E} \left[\int_{t_0}^t c(u, v) d\eta(v) \cdot \overline{\int_{t_0}^t d\eta(v)} \right] = \int_{t_0}^t c(u, v) dF(v). \end{aligned}$$

Elementary computations lead to the equality:

$$B_\eta(s, t) - B_\xi(s, t) = \int_{t_0}^s \int_{t_0}^t K(u, v) dF(u) dF(v),$$

where $K(u, v) = -c(u, v) - \overline{c(v, u)} - \mathbb{E}[a(u) \overline{a(v)}]$.

Thus the condition (3.10) [where $ds dt$ is replaced by $dF(s) dF(t)$] holds true, and the random process $\xi(t)$ is equivalent to $\eta(t)$ if the operator

$$A: \sum c_k \eta(t_k) \rightarrow \sum c_k \xi(t_k)$$

is bounded and invertible. Under the last condition as we know the process $\eta(t)$ is the innovation process for $\xi(t)$, and that condition holds true for instance, if the processes $a(t)$ and $\eta(t)$ are orthogonal to each other. Indeed, in this case the kernel $K(s, t) = -\mathbb{E}[a(s) \overline{a(t)}]$ is negative-definite:

$$\begin{aligned}
 \int_{t_0}^T \int_{t_0}^T K(s, t) u(s) \overline{u(t)} dF(s) dF(t) = \\
 = -E \left| \int_{t_0}^T a(t) u(t) dF(t) \right|^2 < 0
 \end{aligned}$$

for any non-zero function $u(t)$ such that:

$$\int_{t_0}^T |u(t)|^2 dF(t) < \infty.$$

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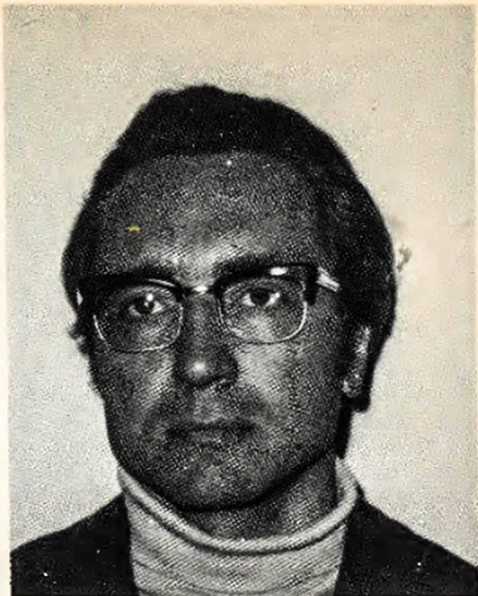
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and estimation of random processes. Based on the author's extensive research, and unifying much scattered material for the first time, the book develops some of the fundamental aspects of Gaussian processes. Intended for students and specialists in probability and functional analysis, *Innovation Processes* is a succinct, systematic reference of value to both pure and applied mathematicians.

SOLUTIONS OF ILL-POSED PROBLEMS

ANDREY N. TIKHONOV and VASILII Y. ARSENIN

Solutions of Ill-Posed Problems is an important contribution to mathematics by two prominent Russian scholars. Academician A. N. Tikhonov, a distinguished Member of the Academy of Sciences of the USSR and Dean at the Moscow State University, is one of the earliest workers in the field of ill-posed problems. His co-author, V. Y. Arsenin, is Professor of Mathematical Physics at the Moscow Institute of Engineering Physics.

The book was written to show that the class of ill-posed problems includes many classical mathematical problems and, most significantly, that such problems have important applications. It defines the concept of an approximate solution of ill-posed problems that is stable to small changes in the initial data and examines in detail methods of constructing solutions that are easily processed on a computer. The reader is introduced to a wide variety of applications in physics, engineering, and automatic processing of observational data. A text for graduate students focused on differential equations, **Solutions of Ill-Posed Problems** is also of special interest to mathematical physicists, applied mathematicians, scientists and engineers interested in prediction of experiments and computer specialists.

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